Euler's Product Formula and the Riemann Zeta Function

David Meyer

dmm@1-4-5.net

Last update: September 20, 2021

1 Introduction

In the 1730s Leonhard Euler proved one of the most important theorems in number theory, known today as Euler's product formula [5]. Euler's product formula exposes the deep relationship between the prime numbers and the Riemann zeta function [8]. Euler's proof appeared in his thesis titled *Variae observationes circa series infinitas* (Various Observations about Infinite Series), which was published by the St. Petersburg Academy in 1737 [4]. The discovery of the product formula was an important advancement in number theory and is perhaps the main reason why (or at least one of the main reasons why) the zeta function plays such a central role in the study of prime numbers.

The Euler product formula for the Riemann zeta function $\zeta(s)$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

Here the left hand side is the famous Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

and the product on the right hand side extends over all prime numbers p:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots$$

So how did Euler find this amazing relationship between the zeta function and the primes?

2 Euler's Proof of the Product Formula

The proof presented here (amazingly) makes use only of simple algebra and is the method by which Euler originally discovered the product formula. Euler's proof is quite clever and takes advantage of a sieving process that is analogous to the ancient Sieve of Eratosthenes [9]. The proof is iterative and proceeds as follows:

Euler's proof starts with the zeta function[1], Equation 1, as the input to the first iteration of the proof.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^2} + \cdots \tag{1}$$

The first step of the algorithm is to multiply both sides of the input, Equation 1, by the second term on its hand right side, $\frac{1}{2^s}$. This gives us

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots \tag{2}$$

The next step is to subtract Equation 2 from Equation 1, which gives us a new equation, Equation 3. This new equation will serve as the input to the next iteration.

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots \tag{3}$$

Notice that this process knocks out all of the terms on the right hand side of the input equation (Equation 1) which have a factor of 2. These are the terms that are "sieved" in this iteration of the algorithm.

Now we repeat the previous steps with their output (Equation 3) as input. Accordingly we multiply Equation 3 by the the second term on its right hand side, $\frac{1}{3^s}$, which gives us

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \cdots \tag{4}$$

Subtracting Equation 4 from Equation 3 we get a new equation, Equation 5:

[1]Euler first studied $\zeta(s)$ as a real function; Riemann was the first to view it as a complex function [2].

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots \tag{5}$$

This iteration knocks out all of the terms on the right hand side of Equation 3 which have a factor of 3. The next iteration will knock out all of the terms on the right hand side that have a factor of 5, yielding

$$\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots$$

What we see is that this algorithm "sieves" the right hand sides of these successive equations while building up the product of $\left(1-\frac{1}{p^s}\right)$ terms for prime p on the left hand side.

If we repeat this process infinitely we wind up with

$$\cdots\left(1-\frac{1}{13^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1$$

Solving for $\zeta(s)$ we get

$$\zeta(s)=\frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{13^s}\right)\cdots}$$

which can be written in a perhaps more familiar (and more compact) way as an infinite product over all of the primes:

$$\zeta(s)=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$

which is Euler's product formula for the zeta function[2].

Finally, to make this proof rigorous we need to show that when $\Re(s)>1$ the right hand side of the sieved equations approaches 1. Fortunately for us this follows immediately from the convergence of the Dirichlet series for $\zeta(s)$ [7]. $\square$

[2] This result is frequently referred to simply as "Euler's product formula".

3 Conclusions

Euler's product formula is one of the most important results in the history of number theory and paved the way for Dirichlet, Riemann and others to the fuse arithmetic and analysis into analytic number theory [6]. In fact, so important is Euler's product formula in the history of number theory that John Derbyshire, in his 2003 book *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* [3], calls Euler's result the "Golden Key" to indicate it's significance in the development of analytic number theory and number theory in general.

As an aside, Mathologer has a very nice video on Euler's product formula that has some fancy animations that make it easy to see how the sieve works [1].

References

[1] Burkard Polster (aka Mathologer). Euler's Pi Prime Product and Riemann's Zeta Function. `https://www.youtube.com/watch?v=LFwSIdLSosI`, Sep 2017. [Online; accessed 10-August-2021].

[2] Dan Nichols. Riemann's Zeta Function and the Prime Number Theorem. `https://people.math.umass.edu/~nichols/talks/2016mathclub-zeta.pdf`, Dec 2016. [Online; accessed 19-September-2021].

[3] John Derbyshire. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. The National Academies Press, Washington, DC, 2003.

[4] J J O'Connor and E F Robertson. A History of the Calculus. `https://mathshistory.st-andrews.ac.uk/HistTopics/The_rise_of_calculus/`, Feb 1996. [Online; accessed 10-August-2021].

[5] Timothy Murphy. Euler's Product Formula. `https://www.maths.tcd.ie/pub/Maths/Courseware/428/Primes-II.pdf`, 2006. [Online; accessed 25-June-2021].

[6] Wikipedia contributors. Analytic Number Theory — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Analytic_number_theory&oldid=1036896346`, 2021. [Online; accessed 8-August-2021].

[7] Wikipedia contributors. Dirichlet Series — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Dirichlet_series&oldid=1016339331`, 2021. [Online; accessed 10-August-2021].

[8] Wikipedia contributors. Riemann Zeta Function — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Riemann_zeta_function&oldid=1036808000`, 2021. [Online; accessed 6-August-2021].

[9] Wikipedia contributors. Sieve of Eratosthenes — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Sieve_of_Eratosthenes&oldid=1037751406, 2021. [Online; accessed 10-August-2021].