What Is The Difference In Our Ages?

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Last update: June 17, 2023

1 Introduction

Suppose your age is x_t and my age is y_t at some time t. Then in n years your age is $x_t + n$ and my age is $y_t + n$. This implies that as time goes by (as measured by $n \in \mathbb{N}$) the difference in our ages vanishes!

Why? Consider that

$$\lim_{n\to\infty}\left[\frac{x_t + n}{y_t + n}\right] = 1$$

which is another way of saying the same thing. This result is reassuring since it pretty much models our common experience. But why is this true? As we will see in Section 2.1, this "ratio of our ages" sequence is a *convergent sequence* in $\mathbb{R}$ with limit one.

1.1 A More General Formulation

More generally consider the function $f_a(n) = a + n$ where $a, n \in \mathbb{N}$. Then

$$\lim_{n\to\infty}\left[\frac{f_a(n)}{f_b(n)}\right] = 1 \tag{1}$$

for $a, b, n \in \mathbb{N}$.

Equation (1) is also frequently written using the following alternate notation:

$$f_a(n) \sim f_b(n)$$

where the $\sim$ symbol means that the ratio of its two arguments tends towards 1 as its arguments tend toward ∞. Said another way:

$$f(x) \sim g(x) \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

2 Proof Using Sequences

Recall our problem setup: Suppose your age at some time t is x_t and my age at that same time is y_t. Then in n years your age will be $x_t + n$ and my age will be $y_t + n$, for $n \in \mathbb{N}$.

Now consider the "ratio of our ages" sequence:

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{x_t + n}{y_t + n}\right)_{n\in\mathbb{N}}$$

The notation $(a_n)_{n\in\mathbb{N}}$ means that a is a map from $\mathbb{N}$ to $\mathbb{R}$, namely

$$a : \mathbb{N} \to \mathbb{R}$$

That is, $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers.

Definition 2.1. Convergent Sequence: We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is *convergent* to $a \in \mathbb{R}$ if for all ϵ greater that zero there exists an N in $\mathbb{N}$ such that if $n \in \mathbb{N}$ is greater than or equal to N then $\mid a_n - a \mid < \epsilon$. Put more concisely

$$\forall \epsilon > 0 \ \ \exists N \in \mathbb{N} \ \ \forall n \geq N : \ \mid a_n - a \mid < \epsilon$$

Here a is called the *limit* of the sequence $(a_n)_{n\in\mathbb{N}_0}$ and we write $\lim\limits_{n\to\infty} a_n = a$ (or sometimes $a_n \xrightarrow{n\to\infty} a$).

So now we want to ask the obvious question: what happens to the ratio of our ages sequence as n goes to ∞? In other words, does this sequence converge, and if it does, to what value?

All of this means that we want to know if this limit

$$\lim_{n\to\infty} \left(\frac{x_t + n}{y_t + n}\right)_{n\in\mathbb{N}}$$

exists and if it does, what is its value.

As we will see in Section 2.1, we do have some machinery we can use to evaluate this limit such as the algebraic rules for limits [1] and the fact that the limit is a linear operator [3].

2.1 Putting It All Together

We can see that the "ratio of our ages" sequence $\left(\dfrac{x_t+n}{y_t+n}\right)_{n\in\mathbb{N}}$ is indeed a convergent sequence with limit equal to one, since

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{x_t+n}{y_t+n}\right) &= \frac{\lim\limits_{n\to\infty}(x_t+n)}{\lim\limits_{n\to\infty}(y_t+n)} && \text{\# quotient rule for limits [1]}\\
&= \frac{\lim\limits_{n\to\infty}(x_t+n)\left(\frac{1}{n}\right)}{\lim\limits_{n\to\infty}(y_t+n)\left(\frac{1}{n}\right)} && \text{\# multiply the fraction by } 1=\tfrac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\\
&= \frac{\lim\limits_{n\to\infty}\left(\dfrac{x_t}{n}+\dfrac{n}{n}\right)}{\lim\limits_{n\to\infty}\left(\dfrac{y_t}{n}+\dfrac{n}{n}\right)} && \text{\# multiply through by } \tfrac{1}{n}\\
&= \frac{\lim\limits_{n\to\infty}\left(\dfrac{x_t}{n}+1\right)}{\lim\limits_{n\to\infty}\left(\dfrac{y_t}{n}+1\right)} && \text{\# } \tfrac{n}{n}=1\\
&= \frac{\lim\limits_{n\to\infty}\dfrac{x_t}{n}+\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}\dfrac{y_t}{n}+\lim\limits_{n\to\infty}1} && \text{\# limit is a linear operator [3]}\\
&= \frac{\lim\limits_{n\to\infty}\dfrac{x_t}{n}+1}{\lim\limits_{n\to\infty}\dfrac{y_t}{n}+1} && \text{\# } \lim_{n\to\infty} c = c \text{ for constant } c \text{ [1]}\\
&= \frac{0+1}{0+1} && \text{\# } \lim_{n\to\infty}\frac{c}{n} = c\lim_{n\to\infty}\frac{1}{n} = 0 \text{ for constant } c \text{ [2]}\\
&= \frac{1}{1} && \text{\# simplify}\\
&= 1 && \text{\# apparently we're all roughly the same age}
\end{aligned}
$$

3 Conclusions

So I guess that while in our short lives our ages appear to vary widely across our life spans, when we look at a bigger picture that difference disappears.

Acknowledgements

LaTeX Source

References

[1] Alex Svirin. Properties of Limits. https://math24.net/properties-limits.html. [Online; accessed 14-November-2021].

[2] Bonnie Saunders. Limits of Sequences. http://homepages.math.uic.edu/~saunders/MATH313/INRA/INRA_Chapter2.pdf, August 2014. [Online; accessed 14-November-2021].

[3] Richard Melrose. Introduction to Functional Analysis. https://ocw.mit.edu/courses/mathematics/18-102-introduction-to-functional-analysis-spring-2009, 2009. [Online; accessed 02-November-2021].