A Note on Algebraic Structures

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1 A Few Algebraic Structures and Their Features

Structure	**ABO**[1]	**Identity**	**Inverse**	**Distributive**[2]	**Commutative**[3]	**Comments**
Semigroup	✓	no	no	N/A	no	$(S, \circ)$
Monoid	✓	✓	no	N/A	no	Semigroup plus identity $\in S$
Group	✓	✓	✓	N/A	no	Monoid plus inverse $\in S$
Abelian Group	✓	✓	✓	N/A	✓$(\circ)$	Commutative group
Ring_+	✓	✓	✓	✓	✓$(+)$	Abelian group under $+$
Ring_*	✓	yes/no	no	✓	no	Monoid under $*$
$\text{Field}_{(+,*)}$	✓	✓$(+,*)$	✓$(+,*)$	✓	✓$(+,*)$	Abelian group under $+$ and $*$
Vector Space	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Field
Module	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Ring

Table 1: A Few Algebraic Structures and Their Features

1.1 Abbreviations

1. **ABO:** Associative Binary Operation
    - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$
    - $x \circ y \in S$ for all $x, y \in S$ (S is closed under $\circ$)
2. **Distributive:** Distributive Property
    - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
    - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$
    - $*$ is *distributive* over $+$ if $*$ is left and right distributive
3. **Commutative:** Commutative Property
    - $x \circ y = y \circ x$ for all $x, y \in S$

2 Notes

- Table 1 implies that F $\subset$ R $\subset$ G $\subset$ M $\subset$ SG.
- Whether or not a ring has a multiplicative identity seems to depend on the field of study.

  In general the definition of a ring R doesn't require a multiplicative inverse in R ($a^{-1} \notin R$ for all $a \in R$) or that multiplication be commutative in R. Specifically: R is an Abelian group under $+$ but we don't require that multiplication be commutative (while $a+b=b+a$ for all $a, b \in R$, we don't require that $ab = ba$ for all $a, b \in R$). These are perhaps the main ways in which a ring differs from a field. In addition, as mentioned above in some cases R need not include a multiplicative identity ($1 \notin R$).
- F $\subset$ VS since the field axioms require a multiplicative inverse (a^{-1}) while vector spaces do not. Fields are also commutative under $*$ and vector spaces are not.
- VS $\subset$ Module since the scalars in a module come from a ring as opposed to a field like we find in vector spaces and F $\subset$ R [1].

LaTeX Source

`https://www.overleaf.com/read/fcfcnyxmgzwv`

References

[1] T. S. Blyth. *Module Theory: An Approach to Linear Algebra.* Oxford University Press, July 1977.