A Few Notes on the Basel Problem

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1 Introduction

The Basel problem [1] is an important problem in number theory [6] that was first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734. The Basel problem so is named for the Swiss city in whose university two of the Bernoulli brothers successively served as professor of mathematics (Jakob, 1687 - 1705, and Johann, 1705 - 1748). Coincidentally Euler was born in Basel.

The Basel problem asks whether the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

has a closed form solution, that is, does it converge to a finite number and if it does, what number does it converge to? An example of a related infinite series that does not converge is the harmonic series [4]:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

Interestingly both of the Bernoulli brothers found proofs that the harmonic series diverges.

The Basel problem resisted solution for some 84 years until the then 26 year old Euler finally solved it. Euler's surprising solution is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Note that both the harmonic series and the Basel problem are instances of the famous Riemann zeta function [7], $\zeta(s)$. In particular, the harmonic series is $\zeta(1)$ and the Basel problem is $\zeta(2)$.

2 Euler's Solution

So how did Euler solve the Basel problem? The summary is that Euler's extraordinary mathematical intuition led him to consider sinthe Maclauren series [8] for $\sin x$ and to compare its coefficients to the coefficients of a polynomial constructed from the zeros of $\sin x$. In particular, he reasoned that the coefficients of the Maclaurin series and the polynomial must be equal for equal powers of x.

This approach seems practical given that we know the Maclaurin series for $\sin x$, we know the zeros of $\sin x$, and because the Fundamental Theorem of Algebra [3] (and more specifically the Weierstrass Factorization Theorem [10]) tells us that we can construct a polynomial for $\sin x$ from its zeros[1].

So how exactly did Euler arrive at his fantastic solution?

2.1 Maclaurin (Taylor) Series for $\sin x$

The story begins with the Maclaurin series [8] for $\sin x$. This series was well-known in Euler's time. Note that in the following discussion of the Maclaurin series for $\sin x$ we are using the $f^{(n)}$ notation for the n^{th} derivative of f, noting that $f^{(0)} = f$.

$$
\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n && \text{\# definition of the Maclaurin series} \\
&= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots && \text{\# expand terms} \\
&= \frac{\sin 0}{0!} x^0 + \frac{\cos 0}{1!} x^1 + \frac{-\sin 0}{2!} x^2 + \frac{-\cos 0}{3!} x^3 + \frac{\sin 0}{4!} x^4 \cdots && \text{\#}\ f(x) = \sin x \\
&= 0 + (\tfrac{1}{1!}) x^1 + 0 + (\tfrac{-1}{3!}) x^3 + 0 + (\tfrac{1}{5!}) x^5 + \cdots && \text{\# the sequence is}\ \{0, 1, 0, -1, ...\} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \text{\# simplify}
\end{aligned}
$$

As mentioned above, one of Euler's key insights was to observe that if we can find a polynomial for $\sin x$ then the coefficients of terms of the same power in the Maclaurin series and the polynomial must equal one another.

So how did Euler find a polynomial for $\sin x$?

[1]The terms "zero" and "root" appear to be used interchangeably in mathematics linterature.

2.2 Euler's Polynomial for $\sin x$

We saw in Section 2 that every non-constant single-variable polynomial with complex coefficients has at least one complex root and that a (possibly infinite) polynomial can be constructed from the linear product of a function's roots. Since we know that the roots of $\sin x$ are $0, \pm\pi, \pm 2\pi, \ldots$ we can directly write down a polynomial $P(x)$ for $\sin x$:

$$\sin x = Ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots$$

The tricky part now is finding A. So how did Euler do it?

Euler noticed that for infinitesimally small values of x $\sin x = x$, that is[2]

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Continuing, we also know that

$$\lim_{x\to 0} \frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots}{x} = (-\pi^2)(-(2\pi)^2)(-(3\pi)^2)\cdots$$

Together these imply that

$$\begin{aligned}
1 &= A(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots && \# \ 1 = \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{P(x)}{x} \\
\Rightarrow & \quad A = \frac{1}{(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots} && \# \text{ solve for } A \\
\Rightarrow & \quad A = \frac{1}{(-\pi^2)(-(2\pi)^2)(-(3\pi)^2)\cdots} && \# \text{ evaluate } A \text{ at the 0 root (set } x=0) \\
\Rightarrow & \quad A = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots} && \# \text{ simplify}
\end{aligned}$$

So now we know that

[2]Note that this limit occurs at the zero root of $\sin x$, that is, $\sin 0 = 0$.

$$
\begin{aligned}
\sin x &= Ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots && \#\ \sin x = P(x) \\
&= \frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots} && \#\ A = ((-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots)^{-1} \\
&= \frac{x(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\cdots}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots} && \#\ \text{multiply like terms} \\
&= x \cdot \left[\frac{x^2-\pi^2}{-\pi^2}\right] \cdot \left[\frac{x^2-4\pi^2}{-4\pi^2}\right] \cdot \left[\frac{x^2-9\pi^2}{-9\pi^2}\right]\cdots && \#\ \text{collect terms, multiply squares} \\
&= x \cdot \left[\frac{\pi^2-x^2}{\pi^2}\right] \cdot \left[\frac{4\pi^2-x^2}{4\pi^2}\right] \cdot \left[\frac{9\pi^2-x^2}{9\pi^2}\right]\cdots && \#\ \text{get rid of } - \text{ in the denominators} \\
&= x \cdot \left[1-\frac{x^2}{\pi^2}\right] \cdot \left[1-\frac{x^2}{4\pi^2}\right] \cdot \left[1-\frac{x^2}{9\pi^2}\right]\cdots && \#\ \text{simplify} \\
&= x \cdot \left[1-\frac{x^2}{(1\pi)^2}\right] \cdot \left[1-\frac{x^2}{(2\pi)^2}\right] \cdot \left[1-\frac{x^2}{(3\pi)^2}\right]\cdots && \#\ \text{put into a convenient form} \\
&= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right) && \#\ \text{Euler's polynomial for } \sin x
\end{aligned}
$$

So we see that Euler's polynomial $P(x)$ for $\sin x$ is

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right) \tag{1}$$

and so

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right)$$

Now, we know that $\frac{P(x)}{x} = \frac{\sin x}{x}$. But we also know that the Maclaurin series for $\frac{\sin x}{x}$ is

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Looking at the coefficients of the x^2 terms of $\frac{\sin x}{x}$ we see that for the Maclauren series we have $-\frac{1}{3!} = -\frac{1}{6}$ and for $\frac{P(x)}{x}$ we have

$$-\left(\frac{1}{1^2\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Since the coefficients of the x^2 terms of the Maclaurin series and the polynomial $P(x)$ must be equal we see that

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Multiplying both sides by $-\pi^2$ we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which amazingly is Euler's solution to the Basel problem.

3 Summary of Euler's Solution of the Basel Problem

To summarize Euler's argument, first recall that Euler knew that the Maclaurin series for the sine function was

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Now, if we divide both sides by x we get[3]

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

[3] Recall that we needed $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ to solve for A in $P(x)$, the polynomial for $\sin x$.

We also know from the Weierstrass Factorization Theorem [10] that every entire function [2] can be represented as a (possibly infinite) product involving its zeroes. So Euler assumed that it must be possible to represent $\sin x$ as an infinite product of linear factors given by its roots. Using this machinery and knowing that the zeros of $\sin x$ occur at $0, \pm\pi, \pm 2\pi, \dots$ we see that

$$\begin{aligned} \frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \end{aligned}$$

Multiplying this product out and collecting the x^2 terms (which we are allowed to do by Newton's Identities [5]), we see by induction that the x^2 coefficient of $\frac{\sin x}{x}$ is

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

We also know that the coefficient of x^2 from Maclaurin series for $\frac{\sin x}{x}$ is

$$-\frac{1}{3!} = -\frac{1}{6}$$

One of Euler's many insights was that the coefficients in both the Maclaurin series and the polynomial for $\frac{\sin x}{x}$ must be equal. So for the x^2 term that means that

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Multiplying both sides of this equation by $-\pi^2$ gives Euler's solution to the Basel problem:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

4 A Few Interesting Consequences

One of the first interesting consequences of Euler's proof is that when we plug $\frac{\pi}{2}$ into the product formula for $\sin x$ (Equation 1) we get the Wallis Product [9]. Why? Well, since $\sin\frac{\pi}{2} = 1$ and with a bit of rearranging we see that

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right) \qquad \# \sin x = P(x) \text{ (Equation 1)}$$

$$= x\prod_{n=1}^{\infty}\left(\frac{n^2\pi^2-x^2}{n^2\pi^2}\right) \qquad \# \text{ get a common denominator, multiply through}$$

$$\Rightarrow \sin\frac{\pi}{2} = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{n^2\pi^2-\frac{\pi^2}{4}}{n^2\pi^2}\right) \qquad \# \text{ set } x = \frac{\pi}{2}$$

$$\Rightarrow 1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{\pi^2\left(n^2-\frac{1}{4}\right)}{\pi^2 n^2}\right) \qquad \# \sin\tfrac{\pi}{2} = 1, \text{ factor out } \pi^2$$

$$\Rightarrow 1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{\left(n^2-\frac{1}{4}\right)}{n^2}\right) \qquad \# \text{ cancel } \pi^2$$

$$\Rightarrow 1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{\left(\frac{4n^2-1}{4}\right)}{n^2}\right) \qquad \# \text{ get a common denominator}$$

$$\Rightarrow 1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right) \qquad \# \text{ multiply right side by } 1 = \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$\Rightarrow \frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right) \qquad \# \text{ multiply both sides by } \frac{2}{\pi}$$

$$\Rightarrow \frac{\pi}{2} = \frac{1}{\prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right)} \qquad \# \text{ take the reciprocal of both sides}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{1}{\left(\frac{4n^2-1}{4n^2}\right)}\right) \qquad \# \; \frac{1}{\prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right)} = \prod_{n=1}^{\infty}\left(\frac{1}{\left(\frac{4n^2-1}{4n^2}\right)}\right)$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right) \qquad \# \; \frac{1}{\left(\frac{4n^2-1}{4n^2}\right)} = \frac{4n^2}{4n^2-1}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{(2n)(2n)}{(2n+1)(2n-1)}\right) \qquad \# \text{ factor numerator and denominator}$$

So we wind up with

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{(2n)(2n)}{(2n-1)(2n+1)} \right) = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdot \left(\frac{6}{5} \cdot \frac{6}{7}\right) \cdot \left(\frac{8}{7} \cdot \frac{8}{9}\right) \cdots$$

that is, the Wallis Product.

5 Conclusions

Today many proofs that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ exist. Many of these proofs are considered, at least by some, to be more rigorous than original Euler's simple and elegant "proof". See for example [11].

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