A Few Notes on Category Theory

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1 Introduction

1.1 What is a Category?

[1]

1.2 Morphisms

1.2.1 A Few Algebraic Structures

Structure	ABO[1]	Identity	Inverse	Distributive[2]	Commutative[3]	Comments
Semigroup	✓	no	no	N/A	no	$(S, \circ)$
Monoid	✓	✓	no	N/A	no	Semigroup plus identity $\in S$
Group	✓	✓	✓	N/A	no	Monoid plus inverse $\in S$
Abelian Group	✓	✓	✓	N/A	✓$(\circ)$	Commutative group
Ring_+	✓	✓	✓	✓	✓$(+)$	Abelian group under $+$
Ring_*	✓	yes/no	no	✓	no	Monoid under $*$
$\text{Field}_{(+,*)}$	✓	✓$(+,*)$	✓$(+,*)$	✓	✓$(+,*)$	Abelian group under $+$ and $*$
Vector Space	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Field
Module	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Ring

Table 1: A Few Algebraic Structures and Their Features

where

1. **ABO:** Associative Binary Operation
   - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$
   - $x \circ y \in S$ for all $x, y \in S$ (S is closed under $\circ$)
2. **Distributive:** Distributive Property
   - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
   - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$
   - $*$ is *distributive* over $+$ if $*$ is left and right distributive
3. **Commutative:** Commutative Property
   - $x \circ y = y \circ x$ for all $x, y \in S$

1.2.2 Notation

Definition 1.1. Morphism: A morphism in a category $\mathcal{C}$ is a mathematical structure that connects two objects $A, B \in \mathcal{C}$. The notation is that for $A, B \in \mathcal{C}$, a morphism f from A to B is denoted as either $f : A \to B$ or $f \in \mathcal{C}(A, B)$. Morphisms are also frequently referred to as "arrows".

1.2.3 Diagrams

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g \circ f"'] & B \arrow[dl, "g"] \\
C &
\end{tikzcd}
$$

Here's a crazy (and beautiful) fact: A group is essentially the same thing as a category that has only one object and in which all the morphisms are isomorphisms.

Ok, but why?

Well, first consider a category $\mathcal{C}$ with just one object. Call that object A. That is, the class $\mathbf{Obj}(\mathcal{C})$ contains only one object, namely, A. Since $\mathcal{C}$ only has one object all of $\mathcal{C}$'s morphisms are in $\mathbf{Hom}_{\mathcal{C}}(A, A)$.

Note that I'll abbreviate $\mathbf{Obj}(\mathcal{C})$ as $\mathcal{C}$ and $\mathbf{Hom}_{\mathcal{C}}(A, B)$ as $\mathcal{C}(A, B)$.

Here the category $\mathcal{C}$ consists of

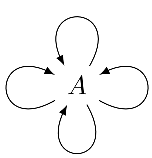
- the class of objects $\mathcal{C}$ consisting of the single object A
- the class of morphisms $\mathcal{C}(A, A)$ consisting of isomorphisms $f : A \to A$
- an associative composition function $\circ : \mathcal{C}(A, A) \times \mathcal{C}(A, A) \to \mathcal{C}(A, A)$
- a two-sided unit 1_A

Interestingly, these four conditions would make $\mathcal{C}(A, A)$ into a group except for inverses. However, saying that every morphism in $\mathcal{C}$ is an isomorphism implies that every element of $\mathcal{C}(A, A)$ has an inverse (with respect to $\circ$). Hence $(\mathcal{C}(A, A), \circ)$ is a group.

So if $(G, \cdot)$ is the group $(\mathcal{C}(A, A), \circ)$ then we have this correspondence:

Category	**Group**
Category $\mathcal{C}$ with single object A	Corresponding group G
Morphisms in $\mathcal{C}$	Elements of G
$\circ \in \mathcal{C}$	$\cdot \in G$
$1_A \in \mathcal{C}$	$1 \in G$

The diagram of $\mathcal{C}$ looks something like



where the arrows represent the different $A \to A$ morphisms in $\mathcal{C}$. These $A \to A$ morphisms are the elements of the group G.

Summary: A group is a category with the special properties that all of the morphisms are invertible and there is only one object.

By a similar argument, a category $\mathcal{C}$ with one object, call it A, is the same thing as a monoid. This is because unlike a group, every element of a monoid is not required to have an inverse. Hence the morphisms in $\mathcal{C}(A, A)$ are not required to be isomorphisms for $\mathcal{C}$ to be a monoid.

1.2.4 Monomorphisms

1.2.5 Epimorphisms

1.2.6 Isomorphisms

1.2.7 Functors

1.3 Natural Transformations

Definition 1.2. Natural Transformation: Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A natural transformation η from F to G is a family of morphisms in D, one for each object A in $\mathcal{C}$, such that

$$\eta_A : F(A) \to G(A)$$

The morphism η_A is called the component of η at object A. In addition, for any morphism $f \in \mathcal{C}(A, B)$ the following diagram must commute:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

Here $F(f)$ and $G(f)$ are the mappings of the morphism $f \in \mathcal{C}(A, B)$ under the functors F and G (respectively), where $F(f) \in \mathcal{D}(F(A), F(B))$ and $G(f) \in \mathcal{D}(G(A), G(B))$. η_A and η_B are natural transformations with $\eta_A \in \mathcal{D}(F(A), G(A))$ and $\eta_B \in \mathcal{D}(F(B), G(B))$.

Commutativity here means that going from $F(A)$ to $G(B)$ via $F(f)$ and η_B is the same as going from $F(A)$ to $G(B)$ via η_A and $G(f)$. More specifically, commutativity requires that for any $f \in \mathcal{C}(A, B)$ we have

$$G(f) \circ \eta_A = \eta_B \circ F(f) \tag{1}$$

Note: I have seen Equation (1) abbreviated as $G\eta = \eta F$.

2 Conclusions

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LaTeX Source

`https://www.overleaf.com/read/wnptmrwwfjgv#a36a79`

References

[1] Introductory Category Theory Notes. Daniel Epelbaum and Ashwin Trisal. https://web.math.ucsb.edu/~atrisal/category%20theory.pdf, July 2020. [Online; accessed 22-Jan-2024].