A Few Notes on Category Theory

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1 Introduction

A category is a collection of objects together with a collection of composable morphisms between these objects. Category theorists use the language "collection of objects" or "collection of morphisms" to sidestep some set-theoretic issues, including: if instead "the set of objects" was used then in the category of sets the set of objects would we be the set of all sets, which we know by Russell's Paradox does not exist [1]. As a result the vague word "collection" might be interpreted as meaning "class" (in the ZF set theory sense [3]).

In any event, a more formal definition of a category is:

Definition 1.1. Category: In category theory, a category is defined by a set of axioms that describe its fundamental properties and behaviors. These are the components of a category:

(i). **Objects**: A category consists of a collection of objects. These objects can be anything: sets, groups, spaces, etc. For example, in the category of sets, the objects are sets themselves. If the collection of objects in a category is a set then the category is called a "small" category.

The collection of objects in a category $\mathcal{C}$ is denoted simply as $\mathcal{C}$, or sometimes $\mathrm{Ob}(\mathcal{C})$. Note that we refer to a generic category as $\mathcal{C}$, while bold font is used to indicate a specific category. For example, the category of sets is called **Set**.

(ii). **Morphisms (Arrows)**: Between every two objects in a category, there exists a collection of morphisms (also called maps or arrows). Morphisms can represent various mathematical structures like functions, homomorphisms, transformations, etc.

The collection of morphisms from A to B in $\mathcal{C}$ is denoted by $\mathcal{C}(A, B)$, or sometimes $\mathrm{Hom}_{\mathcal{C}}(A, B)$. For $A, B \in \mathcal{C}$, a morphism f from A to B is denoted as $f \in \mathcal{C}(A, B)$, $f : A \to B$, or sometimes $A \xrightarrow{f} B$. The latter form is used in commutative diagrams; see Section **??**. All of these forms represent a morphism f with domain A (sometimes denoted at $\mathrm{dom}(f)$) and codomain B (sometimes denoted as $\mathrm{codom}(f)$).

If the collection of morphisms between every pair of objects in a category is a set then the category is called "locally small". Said another way, if for all $A, B \in \mathcal{C}$ the collection of morphisms $\mathcal{C}(A, B)$ is a set then $\mathcal{C}$ is locally small.

(iii). **Identity Morphisms**: For each object A in the category, there exists an identity morphism denoted as id_A or simply 1_A. The identity morphism $\mathrm{id}_A : A \to A$ is the morphism that "does nothing" when composed with other morphisms. It serves as an identity element in the category.

(iv). **Composition of Morphisms**: Morphisms in a category can be composed. If $f : A \to B$ and $g : B \to C$ are morphisms in the category, then there exists a composite morphism denoted as $g \circ f : A \to C$, which represents the composition of f followed by g. One consequence of this requirement: categories are closed under $\circ$.

(v). **Associativity of Composition**: Composition of morphisms is associative. That is, if $f : A \to B$, $g : B \to C$, and $h : C \to D$ are morphisms in the category, then $h \circ (g \circ f) = (h \circ g) \circ f$.

These axioms capture the essential structure of a category and form the basis for studying relationships and structures across various mathematical domains using category theory.

1.1 Morphisms

1.1.1 Commutative Diagrams

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g\circ f"'] & B \arrow[dl, "g"] \\
C &
\end{tikzcd}
$$

1.1.2 Aside: groups, monoids, and categories

Here's a crazy (and beautiful) fact: A group is essentially the same thing as a category that has only one object and in which all the morphisms are isomorphisms.

Ok, but why?

Well, first consider a category $\mathcal{C}$ with just one object. Call that object A. That is, the class $\mathrm{Ob}(\mathcal{C})$ contains only one object, namely, A. Since $\mathcal{C}$ only has one object all of $\mathcal{C}$'s morphisms are in $\mathrm{Hom}_{\mathcal{C}}(A, A)$.

Here the category $\mathcal{C}$ consists of

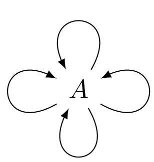
- the class of objects $\mathcal{C}$ consisting of the single object A
- the class of morphisms $\mathcal{C}(A, A)$ consisting of isomorphisms $f : A \to A$
- an associative composition function $\circ : \mathcal{C}(A, A) \times \mathcal{C}(A, A) \to \mathcal{C}(A, A)$
- a two-sided unit 1_A

Interestingly, these four conditions would make $\mathcal{C}(A, A)$ into a group except for inverses. However, saying that every morphism in $\mathcal{C}$ is an isomorphism implies that every element of $\mathcal{C}(A, A)$ has an inverse (with respect to $\circ$). Hence $(\mathcal{C}(A, A), \circ)$ is a group.

So if $(G, \cdot)$ is the group $(\mathcal{C}(A, A), \circ)$ then we have this correspondence:

Category	**Group**
Category $\mathcal{C}$ with single object A	Corresponding group G
Morphisms in $\mathcal{C}$	Elements of G
$\circ \in \mathcal{C}$	$\cdot \in G$
$1_A \in \mathcal{C}$	$1 \in G$

The diagram of $\mathcal{C}$ looks something like



where the arrows represent the different $A \to A$ morphisms in $\mathcal{C}$. These $A \to A$ morphisms are the elements of the group G.

Summary: A group is a category with the special properties that all of the morphisms are invertible and there is only one object.

By a similar argument, a category $\mathcal{C}$ with one object, call it A, is the same thing as a monoid. This is because unlike a group, every element of a monoid is not required to have an inverse (and hence the morphisms in $\mathcal{C}(A, A)$ are not required to be isomorphisms). Here we define the composition $a \circ b$ to be the product ab. Then for $\mathcal{C}$ to be a monoid all that is required is that there be an identity element and that the operation be associative.

1.1.3 Monomorphisms

Before looking at the definition of monomorphism, it is useful to look at the definition of injectivity for sets:

Definition 1.2. Injectivity: A function $f : B \to C$ is said to be injective if for all $b_1, b_2 \in B$ we have

$$f(b_1) = f(b_2) \Rightarrow b_1 = b_2$$

This standard definition of an injective mapping (aka 1:1 mapping) is not suitable for category theory because, among other things, it looks inside the objects. This is an issue because objects in category theory have "nothing inside them", that is, they have no internal structure. In the case of the Definition 1.2, b_1 and b_2 are "inside" the object B.

For functions between sets, injectivity is an important concept. But for maps in an arbitrary category, injectivity might not make sense. So how can we define a similar but more general concept for categories using only morphisms? Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

Since this diagram commutes we know that $f \circ g_1 = f \circ g_2$. f is called a monomorphism if it adheres to Definition 1.3.

Definition 1.3. Monomorphism: Consider a category $\mathcal{C}$ with morphisms $g \in \mathcal{C}(A, B)$ and $f \in \mathcal{C}(B, C)$. Then a morphism $f \in \mathcal{C}(B, C)$ is called a monomorphism if for all $g_1, g_2 \in \mathcal{C}(A, B)$ we have

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2 \tag{1}$$

This property is also called left cancellation.

Although monomorphism is defined solely in terms of morphisms (and not elements), since the above diagram commutes ($f \circ g_1 = f \circ g_2$) both g_1 and g_2 must map to elements in A to the same element in B. This suggests that monomorphism is at least in part about retaining uniqueness (or the lack thereof) in the domain in the codomain (the same is true for injectivity). We can see this since we know by Equation (1) and the law of contraposition [4] that

$$g_1 \neq g_2 \Rightarrow f \circ g_1 \neq f \circ g_2 \tag{2}$$

One way to interpret Equation (2) is that different elements in the domain will be different in the codomain. That is, the uniqueness of g_1 and g_2 is preserved by f. The same kind of argument can be made for injectivity.

Finally, note that while injectivity seems to be the same thing as monomorphism (and injective functions are monomorphisms in **Set**), monomorphism and injectivity are not necessarily the same. Said another way, monomorphism is the generalized-element analogue of injectivity.

1.1.4 Epimorphisms

1.1.5 Isomorphisms

1.1.6 Functors

Example 1.1. Let $\mathcal{C}$ be the category **Set** with $A, B, S \in \mathcal{C}$ and $f : A \to B$. Then there is a functor $- \times S$ defined as follows:

$$- \times S = \begin{cases} A \times S & \text{\# takes } A \text{ to its cartesian product with } S \\ f \times \mathrm{id}_S : A \times S \to B \times S & \text{\# by } (a, s) \mapsto (f(a), s) \end{cases}$$

1.2 Natural Transformations

Definition 1.4. Natural Transformation: Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A natural transformation η from F to G is a family of morphisms in D, one for each object A in $\mathcal{C}$, such that

$$\eta_A : F(A) \to G(A)$$

The morphism η_A is called the component of η at object A. The "naturality condition" is expressed as a commuting square. Specifically, for any $f \in \mathcal{C}(A, B)$ the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

Here $F(f)$ and $G(f)$ are the mappings f under the functors F and G (respectively), where $F(f) \in \mathcal{D}(F(A), F(B))$ and $G(f) \in \mathcal{D}(G(A), G(B))$. η_A and η_B are natural transformations with $\eta_A \in \mathcal{D}(F(A), G(A))$ and $\eta_B \in \mathcal{D}(F(B), G(B))$.

Commutativity here means that going from $F(A)$ to $G(B)$ via $F(f)$ and η_B is the same as going from $F(A)$ to $G(B)$ via η_A and $G(f)$. More specifically, commutativity requires that for any $f \in \mathcal{C}(A, B)$ we have

$$G(f) \circ \eta_A = \eta_B \circ F(f) \tag{3}$$

Note: I have seen Equation (3) abbreviated as $G\eta = \eta F$.

Example 1.2. Consider the functor $- \times S$ defined in Example 1.1. For any function $g : S \to T$

$$\mathrm{id}_{-} \times g : - \times S \to - \times T$$

is a natural transformation. To verify this, pick two objects A, B and a morphism $f : A \to B$. Then we need to show that the following diagram commutes:

$$\begin{tikzcd} A \times S \arrow[r, "f \times \mathrm{id}_S"] \arrow[d, "\mathrm{id}_A \times g"'] & B \times S \arrow[d, "\mathrm{id}_B \times g"] \\ A \times T \arrow[r, "f \times \mathrm{id}_T"'] & B \times T \end{tikzcd}$$

To show that this diagram commutes, pick a pair (a, s) in the top left corner. Then if we go right then down we see that $(a, s) \mapsto (f(a), s) \mapsto (f(a), g(s))$. On the other hand, if we go down then right we see that $(a, s) \mapsto (a, g(s)) \mapsto (f(a), g(s))$. Since both paths between $A \times S$ and $B \times T$ map (a, b) to $(f(a), g(s))$, the diagram commutes and hence $\mathrm{id}_{-} \times g$ a natural transformation [2].

Remark 1.1. On interesting thing is that a natural transformation η kind of commutes the two functors. That is, for functors $F, G : \mathcal{C} \to \mathcal{D}$, if $\eta : F \to G$ is a natural transformation then for any $f \in \mathcal{C}(A, B)$ we have $G(f) \circ \eta_A = \eta_B \circ F(f)$ (Equation (3)). If we drop some detail we see that $G\eta = \eta F$.

2 Conclusions

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LaTeX Source

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Appendix A: A Few Algebraic Structures

Structure	**ABO**[1]	**Identity**	**Inverse**	**Distributive**[2]	**Commutative**[3]	**Comments**
Semigroup	✓	no	no	N/A	no	$(S, \circ)$
Monoid	✓	✓	no	N/A	no	Semigroup plus identity $\in S$
Group	✓	✓	✓	N/A	no	Monoid plus inverse $\in S$
Abelian Group	✓	✓	✓	N/A	✓$(\circ)$	Commutative group
Ring_+	✓	✓	✓	✓	✓$(+)$	Abelian group under $+$
Ring_*	✓	yes/no	no	✓	no	Monoid under $*$
$\text{Field}_{(+,*)}$	✓	✓$(+,*)$	✓$(+,*)$	✓	✓$(+,*)$	Abelian group under $+$ and $*$
Vector Space	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Field
Module	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Ring

Table 1: A Few Algebraic Structures and Their Features

where

1. **ABO:** Associative Binary Operation
    - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$

- $x \circ y \in S$ for all $x, y \in S$ (S is closed under $\circ$)

2. **Distributive:** Distributive Property

   - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
   - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$
   - $*$ is *distributive* over $+$ if $*$ is left and right distributive

3. **Commutative:** Commutative Property

   - $x \circ y = y \circ x$ for all $x, y \in S$