Fermat's Christmas Theorem

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Fermat's Christmas Theorem [1] is a beautiful and simply stated theorem. It is called Fermat's Christmas Theorem because Fermat announced a proof of the theorem in a letter to Mersenne dated December 25, 1640. And of course, Fermat didn't include a proof in the letter.

Fermat's Christmas Theorem (aka Fermat's theorem on sums of two squares) states that an odd prime number p can be expressed as

$$p = r^2 + s^2$$

where $r, s \in \mathbb{N}$, if and only if $p \equiv 1 \pmod 4$.

For example, the primes 5, 13, 17, 29, 37 and 41 are all congruent to 1 modulo 4 and can be expressed as sums of two squares in the following ways:

$$\begin{aligned} 5 &= 1^2 + 2^2 \\ 13 &= 2^2 + 3^2 \\ 17 &= 1^2 + 4^2 \\ 29 &= 2^2 + 5^2 \\ 37 &= 1^2 + 6^2 \\ 41 &= 4^2 + 5^2 \end{aligned}$$

The prime numbers p for which Fermat's Christmas Theorem is true are called Pythagorean primes. See [3] for more on Pythagorean primes.

A variety of proofs of Fermat's Christmas Theorem can be found in [2].

References

[1] Wikipedia contributors. Fermat's theorem on sums of two squares — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Fermat%27s_theorem_on_sums_of_two_squares&oldid=990575323, 2020. [Online; accessed 26-December-2020].

[2] Wikipedia contributors. Proofs of fermat's theorem on sums of two squares — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Proofs_of_Fermat%27s_theorem_on_sums_of_two_squares&oldid=989540767, 2020. [Online; accessed 27-December-2020].

[3] Wikipedia contributors. Pythagorean prime — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Pythagorean_prime&oldid=985939468, 2020. [Online; accessed 26-December-2020].