Christmastime is Christmas Math Time!

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1 Introduction

There are all kinds of interesting mathematics associated with Christmas. This note outlines a few examples, starting with The Infinite Gift and Gabriel's Horn.

2 The Infinite Gift and Gabriel's Horn

Algebraic geometry is populated by all kinds of crazy objects. Being that it is Christmastime (and therefore Christmas Math Time!) please check out the paradoxical object shown in Figure 1. This object is sometimes known as The Infinite Gift.

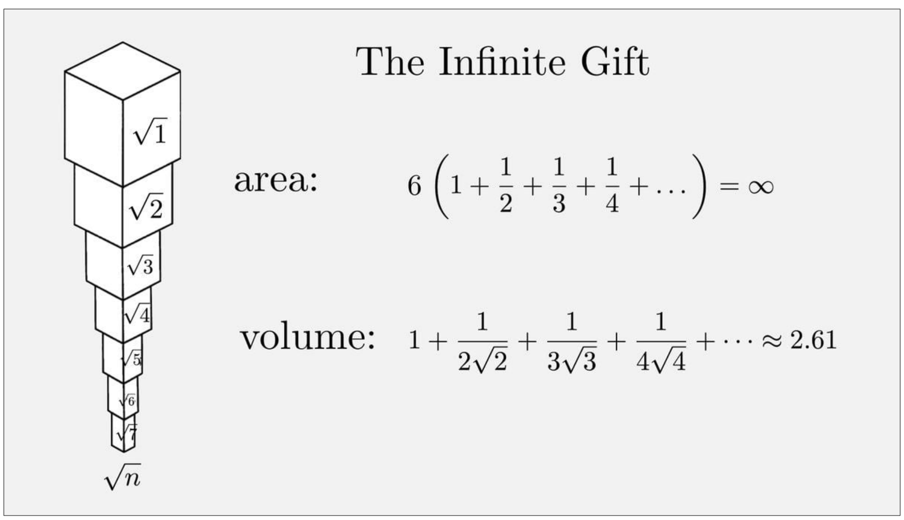


Figure 1: The Infinite Gift

In the Infinite Gift the length of the side of the n^{th} box is $\frac{1}{\sqrt{n}}$ so the area of a side of the n^{th} box is $\left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n}$. Since a box has 6 sides the surface area of the n^{th} box is $6 * \frac{1}{n}$. Then what you find is that in the limit (as $n \to \infty$) the Infinite Gift has infinite surface area but finite volume!

Interesting aside: In the limit the area of the Infinite Gift equals 6 times the harmonic series (which we know diverges [12]).

The continuous version of this object is known as Gabriel's Horn (aka Torricelli's Trumpet) [3], which is shown in Figure 2.

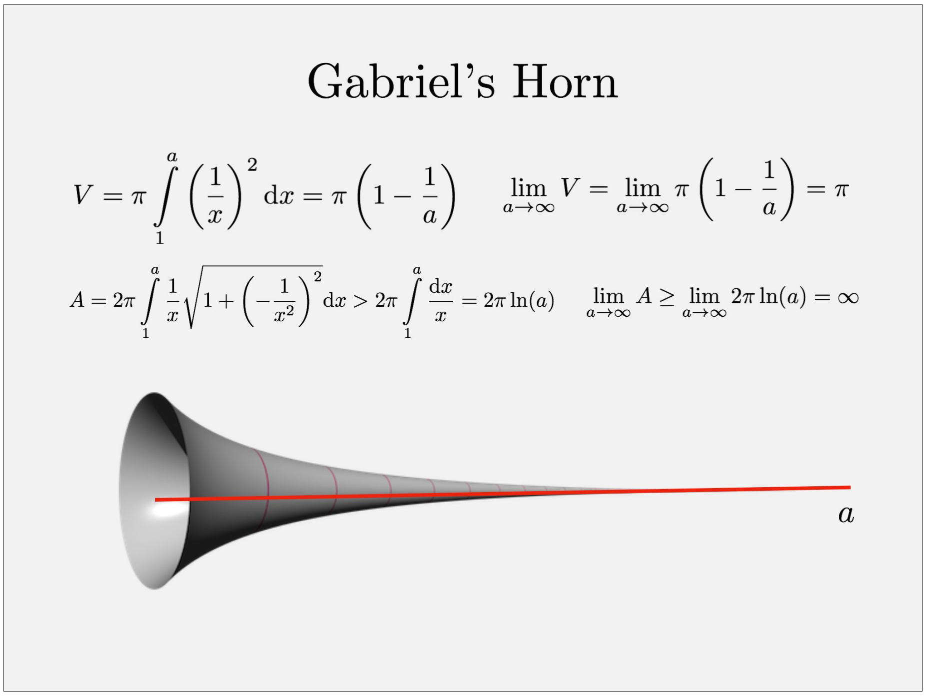


Figure 2: Gabriel's Horn

As we can see in Figure 2, Gabriel's Horn has volume π and area ∞ in the limit. These properties lead to an interesting situation known as the Painter's Paradox [7, 8].

2.1 The Painter's Paradox

This is the Painter's Paradox: Somehow even though you can fill Gabriel's Horn with paint (its volume is finite), you still won't have enough paint to cover its inside surface (its area is infinite)!

3 Fermat's Christmas Theorem

Fermat's Christmas Theorem (aka Fermat's theorem on sums of two squares) is a beautiful and simple theorem which states that an odd prime number p can be expressed as

$$p = r^2 + s^2$$

where $r, s \in \mathbb{N}$, if and only if $p \equiv 1 \pmod 4$. That is, the theorem holds iff $p = 4n + 1$ for some $n \in \mathbb{N}$ [9].

For example, the primes 5, 13, 17, 29, 37 and 41 are all congruent to 1 modulo 4 and can be expressed as sums of two squares in the following ways:

$$\begin{aligned} 5 &= 1^2 + 2^2 \\ 13 &= 2^2 + 3^2 \\ 17 &= 1^2 + 4^2 \\ 29 &= 2^2 + 5^2 \\ 37 &= 1^2 + 6^2 \\ 41 &= 4^2 + 5^2 \end{aligned}$$

On the other hand, the primes 3, 7, 11, 19, 23 and 31 are all congruent to 3 modulo 4 and none of them can be expressed as the sum of two squares. This is the easier part of the theorem since it follows immediately from the observation that all squares are congruent to 0 or 1 modulo 4.

The prime numbers p for which Fermat's Christmas Theorem is true are called Pythagorean primes. See [11] for more on Pythagorean primes.

BTW, this theorem is called Fermat's Christmas Theorem because Fermat announced a proof of the theorem in a letter to Mersenne dated December 25, 1640. And of course, Fermat didn't include a proof in the letter.

A variety of proofs of Fermat's Christmas Theorem can be found in [10].

4 The 12 Days of Christmas

The classic English Christmas carol "The Twelve Days of Christmas" has all kinds of interesting mathematical features. For example, it is an example of a *cumulative song*, in which the lyrics detail a series of increasingly numerous gifts given to the speaker by their "true love" on each of the twelve days of Christmas (the twelve days that make up the Christmas season, starting with Christmas Day) [16]. This cumulative property leads to a bunch of interesting questions including "How many gifts are given each day?" and "What is the total number of gifts given during the 12 Days of Christmas?".

In any event the carol's words were apparently first published in England in the late 18$^{\text{th}}$ century (and has a Roud Folk Song Index number of 68 if you happen to be interested in that kind of thing [18]). A large number of different melodies have been associated with the song, of which the best known is derived from a 1909 arrangement of a traditional folk melody by English composer Frederic Austin [19]. There are twelve verses, each describing a gift given by "my true love" on one of the twelve days of Christmas. Suffice it to say that are many variations in the lyrics. That said, the lyrics I'm using here are from Frederic Austin's 1909 publication that established the current form of the carol [17].

The first three verses of Austin's version of the carol are:

1. On the first day of Christmas my true love sent to me: *A partridge in a pear tree.*
2. On the second day of Christmas my true love sent to me: *Two turtle doves, and a partridge in a pear tree.*
3. On the third day of Christmas my true love sent to me: *Three French hens, two turtle doves, and a partridge in a pear tree.*

The subsequent verses follow the same pattern. The verses deal with the next day where it adds one new gift and then repeats all the earlier gifts. So each verse is one line longer than the previous verse. The subsequent verses add

4. 4 calling birds
5. 5 gold rings
6. 6 geese a-laying
7. 7 swans a-swimming
8. 8 maids a-milking
9. 9 ladies dancing
10. 10 lords a-leaping
11. 11 pipers piping
12. 12 drummers drumming

4.1 How many gifts are received each day and what is the total?

The number of gifts received on day **n** and the total number of gifts received by (including) day **n** are shown in the following table ($1 \leq \mathbf{n} \leq 12$):

Day	Gifts received on day n	Total number of gifts received by day n
1	**1**	**1**
2	1+2=**3**	1+3=**4**
3	1+2+3=**6**	1+3+6=**10**
4	1+2+3+4=**10**	1+3+6+10=**20**
5	1+2+3+4+5=**15**	1+3+6+10+15=**35**
6	1+2+3+4+5+6=**21**	1+3+6+10+15+21=**56**
7	1+2+3+4+5+6+7=**28**	1+3+6+10+15+21+28=**84**
8	1+2+3+4+5+6+7+8=**36**	1+3+6+10+15+21+28+36=**120**
9	1+2+3+4+5+6+7+8+9=**45**	1+3+6+10+15+21+28+36+45=**165**
10	1+2+3+4+5+6+7+8+9+10=**55**	1+3+6+10+15+21+28+36+45+55=**220**
11	1+2+3+4+5+6+7+8+9+10+11=**66**	1+3+6+10+15+21+28+36+45+55+66=**286**
12	1+2+3+4+5+6+7+8+9+10+11+12=**78**	1+3+6+10+15+21+28+36+45+55+66+78=**364**

Table 1: Gift counts for the 12 days of Christmas

Interestingly, the numbers in blue in Table 1, the number of gifts received on day **n**, are called triangular numbers. Why? We will see in Section 4.3 that there is an interesting geometric interpretation that motivates this name.

The n^{th} triangular number, T_n, is defined as follows:

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2} \tag{1}$$

where $\binom{n}{m}$ is the familiar binomial coefficient[1]. For example, $T_{12} = \sum_{k=1}^{12} k = \frac{12 \cdot 13}{2} = \binom{13}{2} = \mathbf{78}$.

So how many total gifts have been received by day **n**? If we look at the **Total number of gifts received by day n** column of Table 1 we can see a pattern. In particular, the number of gifts received by (including) day **n** is the sum of the gifts received on the previous days plus the gifts received on day **n**. The number of gifts received by day **n** is shown in red in Table 1. These numbers are called the tetrahedral numbers and are defined as follows [5]:

$$T_{e_n} = \sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3} \tag{2}$$

where T_k is the k^{th} triangular number (Equation (1)).

It turns out that the total number of gifts received by day 12, $T_{e_{12}}$, is $\sum_{k=1}^{12} T_k = \frac{12 \cdot 13 \cdot 14}{6} = \binom{14}{3} = \mathbf{364}$.

[1] Aside: The great mathematician Carl Friedrich Gauss is said to have discovered the consecutive integer formula (Equation (1)) while still at primary school [1].

BTW, can a number can be both triangular and tetrahedral? Well, if so there would have to exist $m, n \in \mathbb{N}$ such that

$$T_n = \binom{n+1}{2} = \binom{m+2}{3} = T_{e_m} \tag{3}$$

Apparently the only m, n that satisfy Equation (3) are [6]:

$$\begin{aligned}
T_{e_1} &= T_1 &&= 1 \\
T_{e_3} &= T_4 &&= 10 \\
T_{e_8} &= T_{15} &&= 120 \\
T_{e_{20}} &= T_{55} &&= 1540 \\
T_{e_{34}} &= T_{119} &&= 7140
\end{aligned}$$

4.2 Pascal's triangle

"The 12 Days of Christmas" also has interesting connections to Pascal's triangle [13]. Specifically, the 3$^{\text{rd}}$ diagonal of Pascal's triangle, shown in blue in Figure 3, contains the numbers we see in **Gifts received on day n** column of Table 1. These are the triangular numbers (Equation (1)). Similarly, the 4$^{\text{th}}$ diagonal of Pascal's triangle, shown in red in Figure 3, contains the **Total number of gifts received by day n** column of Table 1. These are the tetrahedral numbers (Equation (2)).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

Figure 3: Pascal's triangle with the triangular numbers in blue and the tetrahedral numbers in red

Pascal's triangle can also be drawn with binomial coefficients. This is shown is Figure 4 with the triangular numbers in blue and the tetrahedral numbers in red.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9} \\
\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10} \\
\binom{11}{0} \quad \binom{11}{1} \quad \binom{11}{2} \quad \binom{11}{3} \quad \binom{11}{4} \quad \binom{11}{5} \quad \binom{11}{6} \quad \binom{11}{7} \quad \binom{11}{8} \quad \binom{11}{9} \quad \binom{11}{10} \quad \binom{11}{11} \\
\binom{12}{0} \quad \binom{12}{1} \quad \binom{12}{2} \quad \binom{12}{3} \quad \binom{12}{4} \quad \binom{12}{5} \quad \binom{12}{6} \quad \binom{12}{7} \quad \binom{12}{8} \quad \binom{12}{9} \quad \binom{12}{10} \quad \binom{12}{11} \quad \binom{12}{12} \\
\binom{13}{0} \quad \binom{13}{1} \quad \binom{13}{2} \quad \binom{13}{3} \quad \binom{13}{4} \quad \binom{13}{5} \quad \binom{13}{6} \quad \binom{13}{7} \quad \binom{13}{8} \quad \binom{13}{9} \quad \binom{13}{10} \quad \binom{13}{11} \quad \binom{13}{12} \quad \binom{13}{13} \\
\binom{14}{0} \quad \binom{14}{1} \quad \binom{14}{2} \quad \binom{14}{3} \quad \binom{14}{4} \quad \binom{14}{5} \quad \binom{14}{6} \quad \binom{14}{7} \quad \binom{14}{8} \quad \binom{14}{9} \quad \binom{14}{10} \quad \binom{14}{11} \quad \binom{14}{12} \quad \binom{14}{13} \quad \binom{14}{14}
\end{array}$$

Figure 4: Pascal's triangle with binomial coefficients

4.3 Geometric interpretations

There are interesting geometric interpretations of both the triangular and tetrahedral numbers. So first, what exactly is a triangular number? The first six triangular numbers are depicted as triangles in Figure 5. We can see from the figure that the n^{th} triangular number, T_n, is the sum of the integers from 1 to n [14]. This is the number of gifts received on the n^{th} day of Christmas according to "The 12 Days of Christmas".

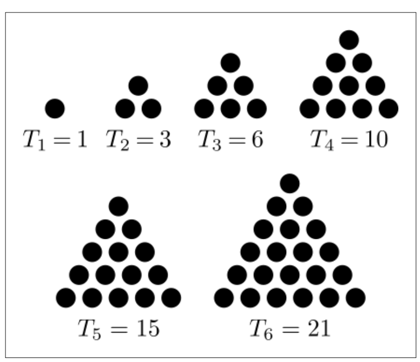


Figure 5: The first six triangular numbers

Ok, then what are the tetrahedral numbers? As we saw in Equation (2), the n^{th} tetrahedral number is the sum of the first n triangular numbers [4]. If we stack up these triangular numbers we get a tetrahedron (hence the name "tetrahedral number"). For example, T_{e_4} is depicted in Figure 6. The n^{th} tetrahedral number turns out to be the number of gifts received up to and including day **n** in "The 12 Days of Christmas".

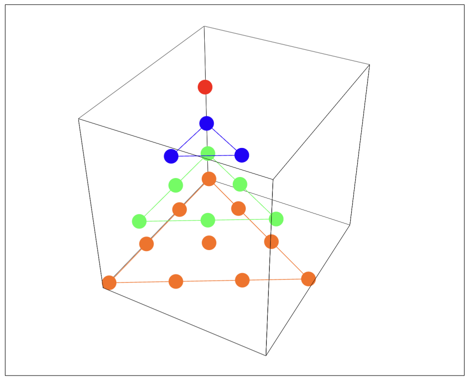


Figure 6: The fourth tetrahedral number $T_{e_4} = 20$

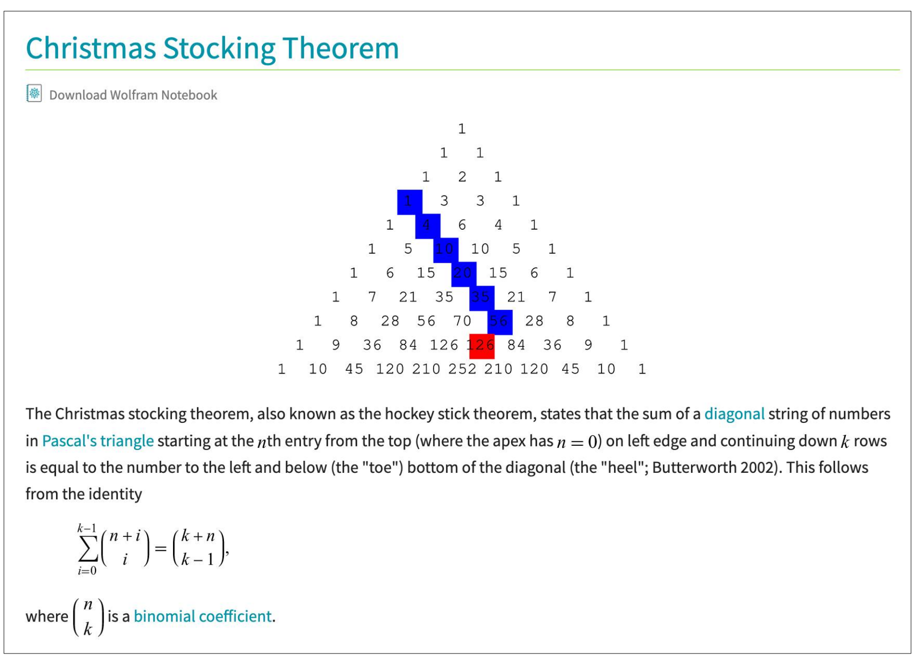
5 The Christmas Stocking Theorem

"The Christmas stocking theorem, also known as the hockey stick theorem, states that the sum of a diagonal string of numbers in Pascal's triangle starting at the nth entry from the top (where the apex has n=0) on left edge and continuing down k rows is equal to the number to the left and below (the "toe") bottom of the diagonal (the "heel"; Butterworth 2002)." [2].

That is, Christmas Stocking Theorem states that for all $n \in \mathbb{N}$ we have

$$\sum_{i=0}^{k-1} \binom{n+i}{i} = \binom{k+n}{k-1}$$

where $\binom{n}{k}$ is a binomial coefficient (aka "n choose k") [15]. This is shown in Figure 7.



Christmas Stocking Theorem

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

The Christmas stocking theorem, also known as the hockey stick theorem, states that the sum of a diagonal string of numbers in Pascal's triangle starting at the nth entry from the top (where the apex has $n = 0$) on left edge and continuing down k rows is equal to the number to the left and below (the "toe") bottom of the diagonal (the "heel"; Butterworth 2002). This follows from the identity

$$\sum_{i=0}^{k-1} \binom{n+i}{i} = \binom{k+n}{k-1},$$

where $\binom{n}{k}$ is a binomial coefficient.

Figure 7: The Christmas Stocking Theorem

6 Merry X-mas!

$$y := \frac{\ln\left(\frac{x}{m} - sa\right)}{r^2} \qquad \text{\# define } y$$

$$\Rightarrow \quad r^2 y = \ln\left(\tfrac{x}{m} - sa\right) \qquad \text{\# multiply both sides on the left by } r^2$$

$$\Rightarrow \quad e^{r^2 y} = e^{\ln\left(\frac{x}{m} - sa\right)} \qquad \text{\# } x = y \Rightarrow b^x = b^y \text{ ; exponentiate with } b = e$$

$$\Rightarrow \quad e^{r^2 y} = \tfrac{x}{m} - sa \qquad \text{\# } e^{\ln(x)} = x$$

$$\Rightarrow \quad e^{r^2 y} = \tfrac{x}{m} - as \qquad \text{\# assume multiplication is commutative } (sa = as)$$

$$\Rightarrow \quad m e^{r^2 y} = x - mas \qquad \text{\# multiply both sides on the left by } m$$

$$\Rightarrow \quad \mathbf{me^{rry} = x - mas} \qquad \text{\# } r^2 = rr \Rightarrow \textbf{Merry X-mas!}$$

Acknowledgements

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