A Curious Integral: $\int\limits_0^1 x \cdot \sqrt{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x \cdots}}}\, dx = \frac{1}{e}$

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Last update: January 19, 2024

1 Ok, but why?

Consider:

$$\begin{aligned}
\int\limits_0^1 x \cdot \sqrt{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x \cdots}}}\, dx &= \int\limits_0^1 x^{\frac{1}{1}} \cdot x^{\frac{1}{1\cdot 2}} \cdot x^{\frac{1}{1\cdot 2\cdot 3}} \cdot x^{\frac{1}{1\cdot 2\cdot 3\cdot 4}} \cdots \; dx && \text{\# radical} \to \text{exponent notation} \\
&= \int\limits_0^1 x^{\left[\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots\right]}\, dx && \text{\# } x^a \cdot x^b = x^{a+b} \\
&= \int\limits_0^1 x^{\left[\sum_{n=1}^{\infty} \frac{1}{n!}\right]}\, dx && \text{\# } \tfrac{1}{1} + \tfrac{1}{1\cdot 2} + \tfrac{1}{1\cdot 2\cdot 3} + \tfrac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots = \sum_{n=1}^{\infty} \tfrac{1}{n!} \\
&= \int\limits_0^1 x^{\left[\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right) - 1\right]}\, dx && \text{\# } \sum_{n=1}^{\infty} \tfrac{1}{n!} = \left(\sum_{n=0}^{\infty} \tfrac{1}{n!}\right) - \frac{1}{0!} = \left(\sum_{n=0}^{\infty} \tfrac{1}{n!}\right) - 1 \\
&= \int\limits_0^1 x^{e-1}\, dx && \text{\# } \sum_{n=0}^{\infty} \tfrac{1}{n!} = e \Rightarrow \left(\sum_{n=0}^{\infty} \tfrac{1}{n!}\right) - 1 = e - 1 \text{ [2]} \\
&= \left.\frac{x^{(e-1)+1}}{(e-1)+1}\right|_0^1 && \text{\# by the power rule [1] and the FToC [3]} \\
&= \left.\frac{x^e}{e}\right|_0^1 && \text{\# } (e-1)+1 = e \\
&= \frac{1^e}{e} - \frac{0^e}{e} && \text{\# evaluate at the endpoints} \\
&= \frac{1}{e} - \frac{0}{e} && \text{\# } 1^e = 1 \text{ and } 0^e = 0 \\
&= \frac{1}{e} && \text{\# } \int\limits_0^1 x \cdot \sqrt{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x \cdots}}}\, dx = \frac{1}{e}
\end{aligned}$$

2 Conclusions

Acknowledgements

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References

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