A Few Notes on the Fourier Series

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1 Introduction

2 Vector Spaces: Linear Algebra vs. the Fourier Series

Here we will use the convention that the constant c denotes the constant function $f(x) = c$, for all x, when the context indicates that c is a function. On the other hand, the notation $c \cdot c$ denotes the scalar multiplication of the scalar value c with itself. Consequentially, I will use 1 to represent $f(x) = 1$ and use context to disambiguate the function $f(x)$ from the scalar value 1. For example, in Equation (1), $\langle 1, 1\rangle = \langle f(x), f(x)\rangle$ (1 represents the constant function $f(x) = 1$), while the notation $1 \cdot 1$ represents the scalar multiplication of the scalar value 1 with itself.

What	Linear Algebra	Fourier Series
Vector Space	$\mathbb{R}^n$	Piecewise smooth 2π-periodic functions on $\mathbb{R}$
Inner Product	$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{n} u_i v_i$	$\langle f(t), g(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$
Orthonormal Basis ($\mathbb{R}^3$)	$\{(1,0,0),(0,1,0),(0,0,1)\}$	$\{1, \cos mt, \sin nt\},\ m,n \in \mathbb{N}\backslash\{0\}$
Representation of a Vector in the Basis	$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{e}_i$	$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$
Coefficients are Projections	$a_i = \langle \mathbf{x}, \mathbf{e}_i\rangle$	$a_0 = \langle f(t), 1\rangle$ $a_m = \langle f(t), \cos mt\rangle$ $b_m = \langle f(t), \sin mt\rangle$

Table 1: Vector Spaces: Linear Algebra vs. Fourier Series

2.1 Orthonormal Bases and the Fourier Series

Some people (specifically Rahul Narain (@narain@mathstodon.xyz)) feel that the orthonormal basis for the Fourier series should be $\{1, \sqrt{2}\cos mt, \sqrt{2}\sin nt\}$, $m, n \in \mathbb{N}\backslash\{0\}$. Ok, but why? Well, notice that $\langle \hat{\mathbf{u}}, \hat{\mathbf{u}}\rangle = 1$ for all unit vectors $\hat{\mathbf{u}}$. However, $\langle 1, 1\rangle = 2$, since

$$\langle 1, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot 1\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} dt = \frac{1}{\pi}\,t\,\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\big(\pi - (-\pi)\big) = \frac{1}{\pi}2\pi = 2 \tag{1}$$

Why does Equation (1) hold? Consider

$$
\begin{aligned}
\langle 1, 1\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)f(t)\,dt && \text{\# since in } 1 = f(t) \text{ and the definition of the inner product (Table 1)}\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} 1\cdot 1\,dt && \text{\# since in } f(t) = 1 \text{ for all } t\in\mathbb{R}\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} dt && \text{\# since } 1\cdot 1 = 1\\
&= \frac{1}{\pi}\,t\Big|_{-\pi}^{\pi} && \text{\# by the FToC [2]}\\
&= \frac{1}{\pi}\big(\pi-(-\pi)\big) && \text{\# evaluate at the end points}\\
&= \frac{1}{\pi}2\pi && \text{\# simplify}\\
&= 2 && \text{\# } \langle 1,1\rangle = 2 \text{ [1]}
\end{aligned}
$$

On the other hand, $\langle \cos nt, \cos nt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 nt\,dt = \frac{1}{\pi}\pi = 1$. In the same way, $\langle \sin nt, \sin nt\rangle = 1$.

We can also see that the vectors in the basis $\{1, \cos mt, \sin nt\}, m, n \in \mathbb{N}\backslash\{0\}$ are orthogonal:

$$\langle 1, \sin nt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nt\,dt = 0$$

Similarly, $\langle 1, \cos nt\rangle = \langle \cos mt, \sin nt\rangle = 0$.

3 Conclusions

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LaTeX Source

`https://www.overleaf.com/read/mtpfwbpmwcpg#b65e5d`

References

[1] Eric Platt. Why are the basis functions of the Fourier series orthogonal? `https://www.quora.com/Why-are-the-basis-functions-of-the-Fourier-series-orthogonal/answer/Eric-Platt-9`, 2020. [Online; accessed 22-December-2023].

[2] Wolfram MathWorld. Fundamental Theory of Calculus. `https://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html`, 2023. [Online; accessed 8-May-2023].