A Few Notes on Functional Analysis

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1 Introduction

Functional Analysis is an interesting topic because, among other reasons, it sits between infinite dimensional linear algebra and real and complex analysis [3]. The connection is the norm $\|\mathbf{x}\|$ of a vector space V. As we shall see, if $(V,\|\cdot\|)$ is a *normed space* then (V, d) is a *metric space* with metric $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$. Moreover, if $(V, \langle\cdot,\cdot\rangle)$ is an *inner product space*[1] then $(V,\|\cdot\|)$ is a normed space with norm $\|\mathbf{x}\| := \sqrt{\langle\mathbf{x}, \mathbf{x}\rangle}$ and consequently (V, d) is a metric space with metric $d(\mathbf{x}, \mathbf{y}) = \sqrt{\langle\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}\rangle}$.

One quick thing is to notice is that as we go from metric to norm to inner product we are adding structure to a vector space. Specifically

$$
\begin{array}{rccl}
\text{metric:} & d(x,y) & \longrightarrow & \text{measures distances} \\
\text{norm:} & \|\mathbf{x}\| & \longrightarrow & \text{measures distances, lengths} \\
\text{inner product:} & \langle\mathbf{x},\mathbf{y}\rangle & \longrightarrow & \text{measures distances, lengths and angles}
\end{array}
$$

If we consider an arbitrary set X with no other structure we can only tell if two points are equal. A metric adds structure to X which allows us to calculate the distance between any two points $x, y \in X$. Similarly, a norm adds structure to a vector space that allows us to calculate length (as well as distance and equality). An inner product adds the ability to calculate angles, etc. Both the algebraic and geometric aspects of this increasing structure are pretty interesting.

2 Metric Spaces

Metric spaces, along with vector spaces are among the most fundamental objects in Functional Analysis [2]. The setup we consider is that we have an arbitrary set X and we look at two points $x, y \in X$. What we would like to do is measure the distance between x and y in X as shown in Figure 1.

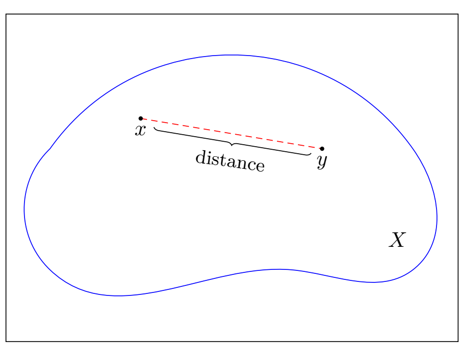

Figure 1: The distance between x and y in X

[1]That is, V is a vector space and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is an inner product on the vector space V.

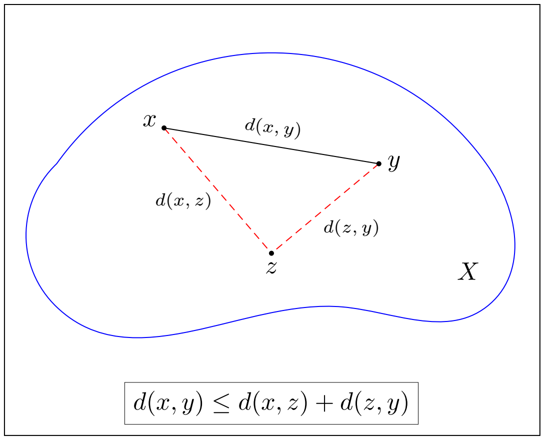

Figure 2: The triangle inequality for distances

To measure this distance we define a map $d : X \times X \to [0, \infty)$, which we call a metric and which has the following three properties:

1. $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$ # this property is called "positive definiteness"
2. $d(x,y) = d(y,x)$ # this property is called "symmetry"
3. $d(x,y) \leq d(x,z) + d(z,y)$ # this is called the "triangle inequality"

We want the distance between x and itself to be zero, hence property 1. However, more than that we want this only to be the case when $x = y$ (so if and only if). Of course we want the distance from x to y to be the same as the distance from y to x; this is property 2. Finally, property 3, the triangle inequality, is what we would expect in normal Euclidian space [1]. The triangle inequality tells us that if we take a "detour" through a third point z then the distance between x and y though z will be greater than (or possibly equal to) the distance between x and y on the direct path. This is shown in Figure 2. Note that there are important differences (which we will get to later) between distance, as measured by a metric $d(x,y)$ and length, as measured by a norm $\|\mathbf{x}\|$, but for now we'll consider the "distance version" of the triangle inequality as shown in Figure 2.

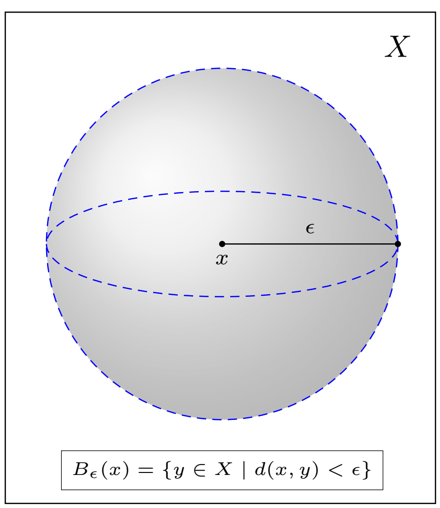

Figure 3: $B_\epsilon(x)$ is an open epsilon ball centered at x with radius ϵ

3 Conclusions

Acknowledgements

LaTeX Source

https://www.overleaf.com/read/fgrrxmkycvry

References

[1] Frank Jones. Euclidean Space. http://www.owlnet.rice.edu/~fjones/chap1.pdf, August 2004. [Online; accessed 03-November-2021].

[2] John O'Connor. Definition and examples of metric spaces. http://www-groups.mcs.st-andrews.ac.uk/~john/MT4522/Lectures/L5.html, February 2004. [Online; accessed 03-November-2021].

[3] Richard Melrose. Introduction to Functional Analysis. https://ocw.mit.edu/courses/mathematics/18-102-introduction-to-functional-analysis-spring-2009, 2009. [Online; accessed 02-November-2021].