

A Few Notes on Functional Analysis

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1 Introduction

Functional Analysis is an interesting topic because, among other reasons, it sits between infinite dimensional linear algebra and real and complex analysis [3]. The connection is the norm $\|\mathbf{x}\|$ of a vector space X . As we shall see, if $(X, \|\cdot\|)$ is a *normed space* then (X, d) is a *metric space* with metric $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$. Moreover, if $(X, \langle \cdot, \cdot \rangle)$ is an *inner product space*¹ then $(X, \|\cdot\|)$ is a normed space with norm $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ and consequently (X, d) is a metric space with metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}$.

A metric $d(\mathbf{x}, \mathbf{y})$ induces a topology τ_d on a set X , where $\tau_d := \{A \subseteq X \mid \text{for every } \mathbf{x} \in A \text{ there exists an } \epsilon \text{ such that } B_\epsilon(\mathbf{x}) \subseteq A\}$ ². That is, τ_d is a collection of open subsets of X . The pair (X, τ_d) is called a *topological space*.

One quick thing to notice is that as we go from metric to norm to inner product we are adding structure to a set or vector space. More specifically

metric:	$d(\mathbf{x}, \mathbf{y})$	\longrightarrow	measures distances
norm:	$\ \mathbf{x}\ $	\longrightarrow	measures distances, lengths
inner product:	$\langle \mathbf{x}, \mathbf{y} \rangle$	\longrightarrow	measures distances, lengths and angles

If we consider an arbitrary set X with no other structure we can only tell if two points are equal. A metric adds structure to X which allows us to calculate the distance between any two points $x, y \in X$. Similarly, a norm adds structure to a vector space that allows us to calculate length (as well as distance and equality). An inner product adds the ability to calculate angles, etc. Both the algebraic and geometric aspects of this increasing structure are pretty interesting.

2 Metric Spaces

Metric spaces, along with vector spaces are among the most fundamental objects in Functional Analysis [2]. The setup we consider is that we have an arbitrary set X and we look at two points $x, y \in X$. What we would like to do is measure the distance between x and y in X as shown in Figure 3.

To measure this distance we define a map $d : X \times X \rightarrow [0, \infty)$, which we call a metric and which has the following three properties:

¹For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the inner product is given by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^n x_i y_i$.

² $B_\epsilon(\mathbf{x})$ is the open epsilon ball centered at \mathbf{x} . See Figure 1.

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ # this property is called *positive definiteness*
2. $d(x, y) = d(y, x)$ # this property is called *symmetry*
3. $d(x, y) \leq d(x, z) + d(z, y)$ # this property is called the *triangle inequality*

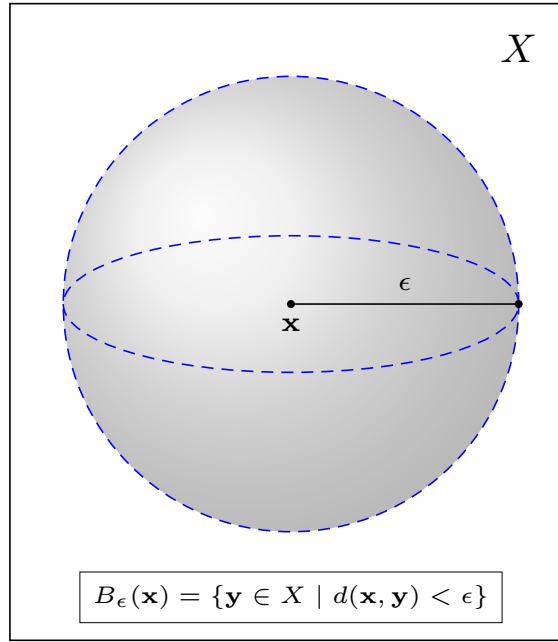


Figure 1: $B_\epsilon(\mathbf{x})$ is an open epsilon ball centered at \mathbf{x} with radius ϵ

We want the distance between x and itself to be zero, hence property 1. However, more than that we want this only to be the case when $x = y$ (so if and only if). Of course we want the distance from x to y to be the same as the distance from y to x ; this is property 2. Finally, property 3, the triangle inequality, is what we would expect in normal Euclidian space [1]. The triangle inequality tells us that if we take a "detour" through a third point z then the distance between x and y though z will be greater than (or possibly equal to) the distance between x and y on the direct path. This is shown in Figure 4. Note that there are important differences (which we will get to later) between distance, as measured by a metric $d(x, y)$ and length, as measured by a norm $\|\mathbf{x}\|$, but for now we'll consider the "distance version" of the triangle inequality as shown in Figure 4.

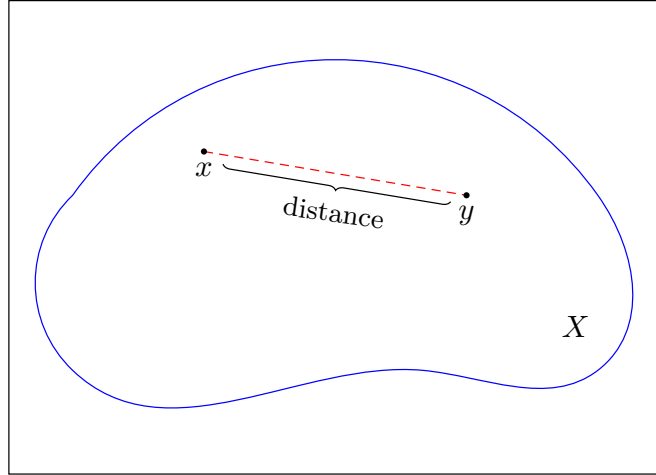


Figure 2: The distance between x and y in X

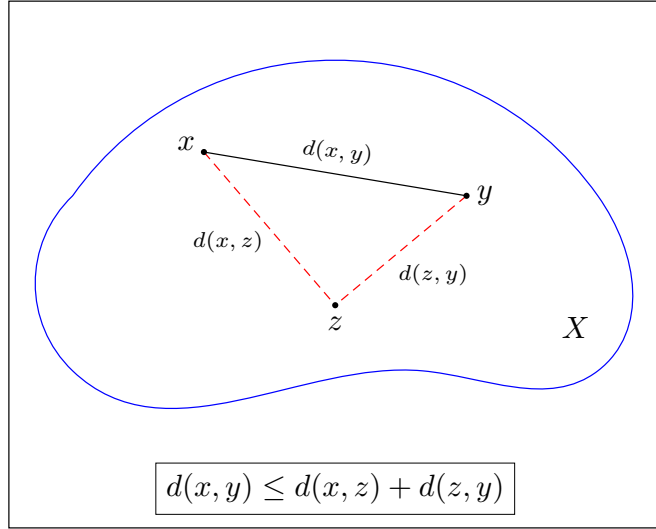


Figure 4: The triangle inequality for distances

3 Conclusions

One way to look at the relationship between inner product, metric, normed, and topological spaces is shown in Figure 5.

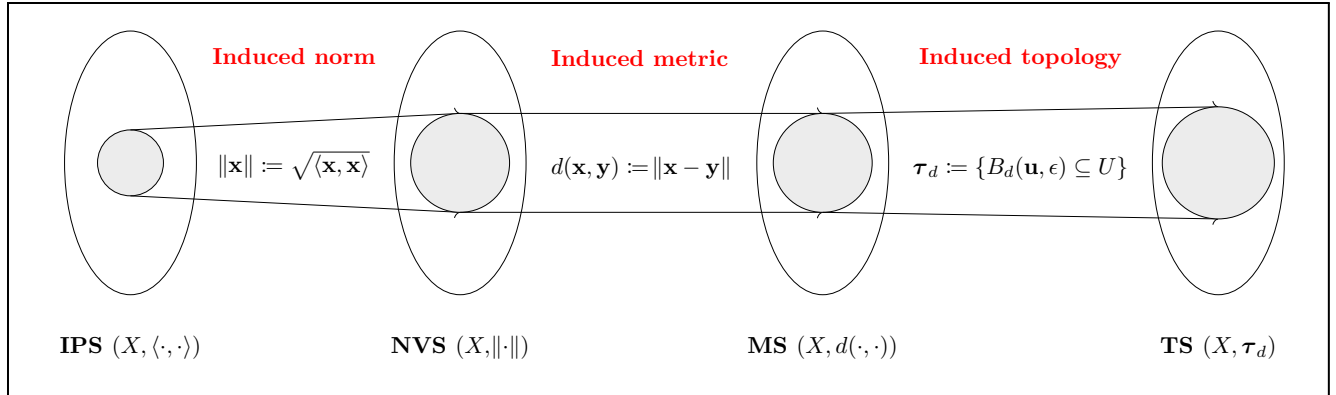


Figure 5: Induced Norm, Metric, and Topology

IPS	--	Inner Product Spaces	$(X, \langle \mathbf{x}, \mathbf{y} \rangle)$
NVS	--	Normed Vector Spaces	$(X, \ \mathbf{x}\) = (X, \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle})$
MS	--	Metric Spaces	$(X, d(\mathbf{x}, \mathbf{y})) = (X, \ \mathbf{x} - \mathbf{y}\) = (X, \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle})$
TS	--	Topological Spaces	(X, τ_d) , where $\tau_d = \{\text{open subsets of } (X, d)\}$

Acknowledgements

L^AT_EX Source

<https://www.overleaf.com/read/fgrrxmkycvry>

References

- [1] Frank Jones. Euclidean Space. <http://www.owl.net.rice.edu/~fjones/chap1.pdf>, August 2004. [Online; accessed 03-November-2021].
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- [3] Richard Melrose. Introduction to Functional Analysis. <https://ocw.mit.edu/courses/mathematics/18-102-introduction-to-functional-analysis-spring-2009>, 2009. [Online; accessed 02-November-2021].

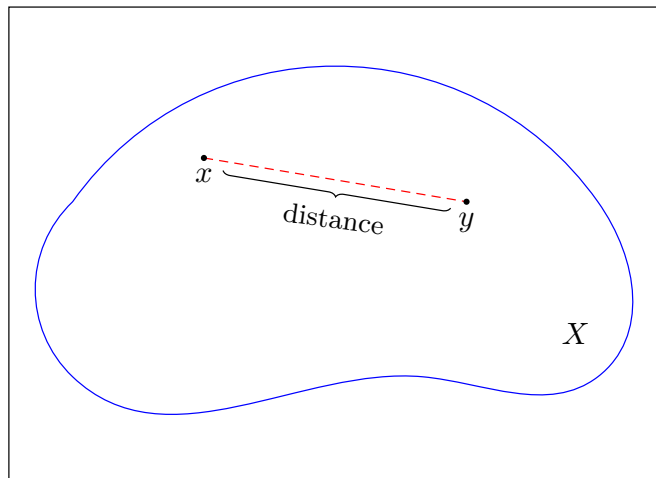


Figure 3: The distance between x and y in X