A Few Notes on Functional Analysis

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Last update: November 3, 2021

1 Introduction

Functional Analysis is an interesting topic because, among other reasons, it sits between infinite dimensional linear algebra and real and complex analysis [3]. The connection is the norm $\|\mathbf{x}\|$ of a vector space V. As we shall see, if $(V, \|\cdot\|)$ is a *normed space* then (V, d) is a *metric space* with metric $d(\mathbf{x}, \mathbf{y}) \coloneqq \|\mathbf{x} - \mathbf{y}\|$. Moreover, if $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space* (that is, $\langle \cdot, \cdot \rangle$ is an inner product on vector space V) then $(V, \|\cdot\|)$ is a normed space with norm $\|\mathbf{x}\| \coloneqq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ and consequently (V, d) is a metric space with metric $d(\mathbf{x}, \mathbf{y}) = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}$.

One quick thing is to notice is that as we go from metric to norm to inner product we are adding structure to a vector space. Specifically

$$
\begin{array}{rccl}
\text{metric:} & d(x, y) & \longrightarrow & \text{measures distances} \\
\text{norm:} & \|\mathbf{x}\| & \longrightarrow & \text{measures distances, lengths} \\
\text{inner product:} & \langle \mathbf{x}, \mathbf{y} \rangle & \longrightarrow & \text{measures distances, lengths and angles}
\end{array}
$$

If we consider an arbitrary set X with no other structure we can only tell if two points are equal. A metric adds structure to such a set which allows us to calculate the distance between any two points (as well as whether they are equal or now). Similarly, a norm adds structure to a vector space that allows us to calculate length (as well as distance and equality). A inner product adds the ability to calculate angles, etc. Both the algebraic and geometric aspects of this increasing structure are pretty interesting.

2 Metric Spaces

Metric spaces, along with vector spaces are among the most fundamental objects in Functional Analysis [2]. The setup we consider is that we have an arbitrary set X and we look at two points $x, y \in X$. What we would like to do is measure the distance between x and y in X as shown in Figure 1.

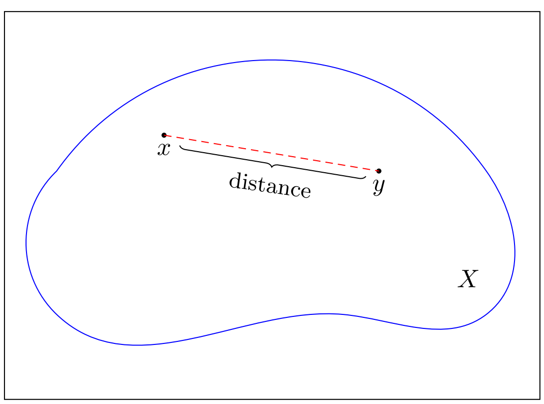

Figure 1: The distance between x and y in X

To measure this distance we define a map $d : X \times X \to [0, \infty)$, which we call a metric and which has the following three properties:

1. $d(x, y) = 0 \Leftrightarrow x = y$ # this property is called "positive definiteness"
2. $d(x, y) = d(y, x)$ # this property is called "symmetry"
3. $d(x, y) \leq d(x, z) + d(z, y)$ # this is called the "triangle inequality"

We want the distance between x and itself to be zero, hence property 1. However, more than that we want this only to be the case when $x = y$ (so if and only if). Of course we want the distance from x to y to be the same as the distance from y to x; this is property 2. Finally, property 3, the triangle inequality, is what we would expect in normal Euclidian space [1]. The triangle inequality tells us that if we take a "detour" through a third point z then the detour path will have a greater or possibly equal than the direct path between x and y (see Figure 2). Note that there are important differences (which we will get to later) between distance, as measured by a metric $d(x, y)$ and length, as measured by a norm $\|\mathbf{x}\|$, but for now we'll consider the "distance version" of the triangle inequality as shown in Figure 2.

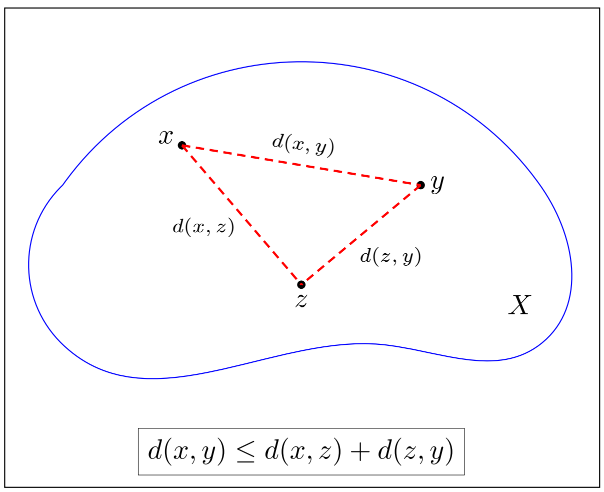

Figure 2: The Triangle Inequality

3 Conclusions

Acknowledgements

LaTeX Source

https://www.overleaf.com/read/fgrrxmkycvry

References

[1] Frank Jones. Euclidean space. http://www.owlnet.rice.edu/~fjones/chap1.pdf, Aug 2004. [Online; accessed 03-November-2021].

[2] John O'Connor. Definition and examples of metric spaces. http://www-groups.mcs.st-andrews.ac.uk/~john/MT4522/Lectures/L5.html, Feb 2004. [Online; accessed 03-November-2021].

[3] Richard Melrose. Introduction to Functional Analysis. https://ocw.mit.edu/courses/mathematics/18-102-introduction-to-functional-analysis-spring-2009, 2009. [Online; accessed 02-November-2021].

Figures

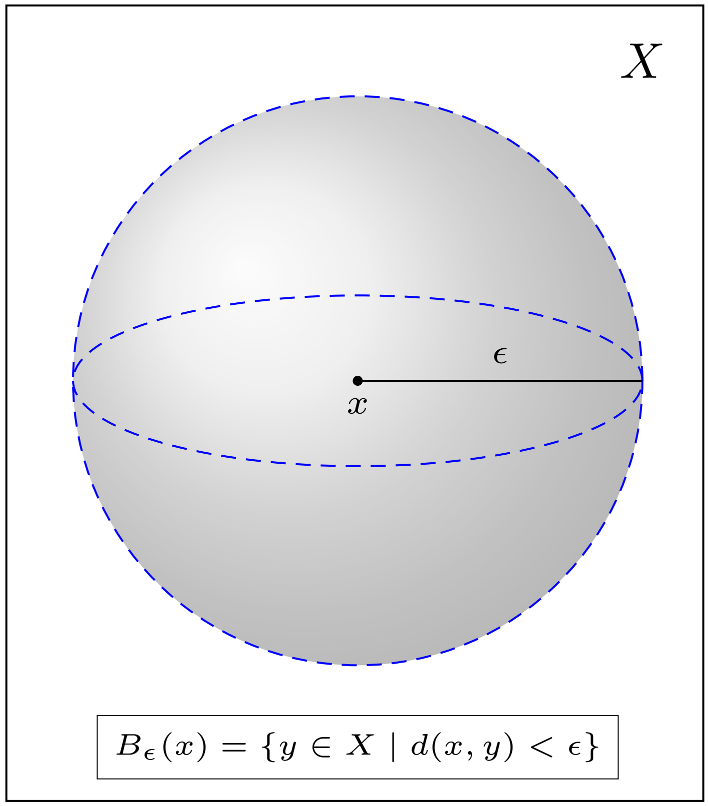

Figure 3: $B_\epsilon(x)$ is an open epsilon ball centered at x with radius ϵ