# An Interesting Integral Involving The Golden Ratio $\phi$

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Consider the following equality:

$$\ln \phi = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{x^2+1}}\, dx \tag{1}$$

Why does Equation (1) hold? To see why, first parameterize with $x = \tan y$ and $\sqrt{x^2+1} = \sec y$. This parameterization means that $dx = \sec^2 y\, dy$, $y = \arctan x$ and $\sec y = \sec(\arctan x) = \sqrt{x^2+1}$. Now we can see that

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x^2+1}}\, dx &= \int_0^{\frac{1}{2}} \frac{\sec^2 y}{\sec y}\, dy && \text{\# use above parameterization} \\
&= \int_0^{\frac{1}{2}} \sec y\, dy && \text{\#}\ \frac{\sec^2 y}{\sec y} = \sec y \\
&= \int_{\arctan 0}^{\arctan \frac{1}{2}} \sec y\, dy && \text{\# limits of integration: } y = \arctan x \\
&= \int_0^{\arctan \frac{1}{2}} \sec y\, dy && \text{\#}\ \arctan 0 = 0 \\
&= \ln|\sec y + \tan y|\Big|_0^{\arctan \frac{1}{2}} && \text{\# integral of } \sec y\ dy \text{ [1] and the FToC} \\
&= \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - \ln|\sec 0 + \tan 0| && \text{\# expand previous line} \\
&= \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - \ln|1+0| && \text{\#}\ \sec 0 = 1 \text{ and } \tan 0 = 0 \\
&= \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - 0 && \text{\#}\ \ln|1+0| = \ln 1 = 0 \\
&= \ln|\sec(\arctan \tfrac{1}{2}) + \tfrac{1}{2}| && \text{\#}\ \tan(\arctan x) = x \text{ and } x - 0 = x \\
&= \ln\left|\sqrt{\left(\tfrac{1}{2}\right)^2 + 1} + \tfrac{1}{2}\right| && \text{\#}\ \sec(\arctan x) = \sqrt{x^2+1} \\
&= \ln\left|\sqrt{\tfrac{1}{4} + 1} + \tfrac{1}{2}\right| && \text{\# arithmetic: } \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4} \\
&= \ln\left|\sqrt{\tfrac{5}{4}} + \tfrac{1}{2}\right| && \text{\# arithmetic: } \sqrt{\tfrac{1}{4}+1} = \sqrt{\tfrac{1}{4}+\tfrac{4}{4}} = \sqrt{\tfrac{5}{4}} \\
&= \ln\left|\tfrac{1+\sqrt{5}}{2}\right| && \text{\# arithmetic: } \sqrt{\tfrac{5}{4}} + \tfrac{1}{2} = \tfrac{\sqrt{5}}{2} + \tfrac{1}{2} = \tfrac{1+\sqrt{5}}{2} \\
&= \ln \phi && \text{\#}\ \phi = \tfrac{1+\sqrt{5}}{2}
\end{aligned}$$

## References

[1] MIT. Integral of Secant. https://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/unit-4-techniques-of-integration/part-a-trigonometric-powers-trigonometric-substitution-and-completing-the-square/session-71-integrals-involving-secant-cosecant-and-cotangent/MIT18_01SCF10_Ses71c.pdf, September 2010. [Online; accessed 23-January-2022].