An Interesting Integral Involving The Golden Ratio ϕ

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Consider the following integral:

$$\ln \phi = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{x^2+1}}\, dx \tag{1}$$

To see how Equation (1) works, first parameterize it with $x = \tan y$ and $\sqrt{x^2+1} = \sec y$. Then $dx = \sec^2 y\, dy$, $y = \arctan x$, and $\sec y = \sec(\arctan x) = \sqrt{x^2+1}$. So

$$\begin{array}{rcll}
\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x^2+1}}\, dx & = & \displaystyle\int_0^{\frac{1}{2}} \frac{\sec^2 y}{\sec y}\, dy & \#\ \text{use above parameterization} \\
& = & \displaystyle\int_0^{\frac{1}{2}} \sec y\; dy & \#\ \dfrac{\sec^2 y}{\sec y} = \sec y \\
& = & \displaystyle\int_{\arctan 0}^{\arctan \frac{1}{2}} \sec y\; dy & \#\ \text{parameterization} \Rightarrow y = \arctan x \\
& = & \displaystyle\int_0^{\arctan \frac{1}{2}} \sec y\; dy & \#\ \arctan 0 = 0 \\
& = & \ln|\sec y + \tan y|\Big|_0^{\arctan \frac{1}{2}} & \#\ \text{integral of } \sec y\; dy \text{ [1] and the FToC [4]} \\
& = & \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - \ln|\sec 0 + \tan 0| & \#\ \text{expand previous line} \\
& = & \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - \ln|1+0| & \#\ \sec 0 = 1 \text{ and } \tan 0 = 0 \\
& = & \ln|\sec(\arctan \tfrac{1}{2}) + \tan(\arctan \tfrac{1}{2})| - 0 & \#\ \ln|1+0| = \ln 1 = 0 \\
& = & \ln|\sec(\arctan \tfrac{1}{2}) + \tfrac{1}{2}| & \#\ \tan(\arctan x) = x \\
& = & \ln\left|\sqrt{\left(\tfrac{1}{2}\right)^2 + 1} + \tfrac{1}{2}\right| & \#\ \sec(\arctan x) = \sqrt{x^2+1} \\
& = & \ln\left|\sqrt{\tfrac{1}{4} + 1} + \tfrac{1}{2}\right| & \#\ \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4} \\
& = & \ln\left|\sqrt{\tfrac{5}{4}} + \tfrac{1}{2}\right| & \#\ \sqrt{\tfrac{1}{4}+1} = \sqrt{\tfrac{1}{4}+\tfrac{4}{4}} = \sqrt{\tfrac{5}{4}} \\
& = & \ln\left|\tfrac{1+\sqrt{5}}{2}\right| & \#\ \sqrt{\tfrac{5}{4}} + \tfrac{1}{2} = \tfrac{\sqrt{5}}{2} + \tfrac{1}{2} = \tfrac{1+\sqrt{5}}{2} \\
& = & \ln \phi & \#\ \phi = \tfrac{1+\sqrt{5}}{2} \text{ [2]}
\end{array}$$

Acknowledgements

Paul Masson (@paulmasson@mathstodon.xyz) pointed out that a faster way to get the result is to recognize that the integral is the inverse hyperbolic sine and then use its logarithmic form [5]. So for $x \in \mathbb{R}$ we have:

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{2}$$

Next, notice that

$$\ln\left(x + \sqrt{x^2+1}\right) = \ln c \Rightarrow x + \sqrt{x^2+1} = c \tag{3}$$

Then the upper limit of integration for the integral in Equation (2) in terms of c is

$$\begin{aligned}
x + \sqrt{x^2+1} &= c && \text{\# Equation (3)} \\
&\Rightarrow x^2 + 2x\sqrt{x^2+1} + x^2 + 1 = c^2 && \text{\# square both sides} \\
&\Rightarrow 2x^2 + 2x\sqrt{x^2+1} + 1 = c^2 && \text{\# collect terms} \\
&\Rightarrow 2x^2 + 2x\sqrt{x^2+1} = c^2 - 1 && \text{\# subtract 1 from both sides} \\
&\Rightarrow 2x(x + \sqrt{x^2+1}) = c^2 - 1 && \text{\# factor out } 2x \\
&\Rightarrow 2xc = c^2 - 1 && \text{\# } x + \sqrt{x^2+1} = c \\
&\Rightarrow x = \frac{c^2-1}{2c} && \text{\# solve for } x
\end{aligned}$$

So now we know that

$$\int_0^{\frac{c^2-1}{2c}} \frac{1}{\sqrt{x^2+1}}\,dx = \ln c \tag{4}$$

Equation (4) holds for $c \in \mathbb{Z}, \mathbb{R}, \mathbb{C}, \ldots$ [@paulmasson@mathstodon.xyz].

If we set $x = \frac{1}{2}$ in Equation (3) then $c = x + \sqrt{x^2+1} = \frac{1}{2} + \sqrt{\frac{5}{4}} = \frac{1+\sqrt{5}}{2} = \phi$. Alternatively, we can see that $c = \phi$ when the upper limit of integration in Equation (2) equals $\frac{1}{2}$, since

$$\begin{aligned}
\frac{c^2-1}{2c} &= \frac{1}{2} && \text{\# set the upper limit of integration } \left(\tfrac{c^2-1}{2c}\right) \text{ to } \tfrac{1}{2} \\
&\Rightarrow \frac{c^2-1}{c} = 1 && \text{\# multiply both sides by 2} \\
&\Rightarrow c^2 - 1 = c && \text{\# multiply both sides by } c \\
&\Rightarrow c^2 - c - 1 = 0 && \text{\# } c^2 - c - 1 \text{ is } \phi\text{'s minimal polynomial}
\end{aligned}$$

Here we can conclude that $c = \phi$, since $c^2 - c - 1$ is ϕ's minimal polynomial and thus has ϕ as it's positive root [2]. Checking this numerically we see that

$$
\begin{aligned}
c^2 - c - 1 &= 0 && \text{\# } \phi\text{'s minimal polynomial} \\
&\Rightarrow \quad c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} && \text{\# solve for } c \text{ using the quadric formula [3]} \\
&\Rightarrow \quad c = \frac{1 \pm \sqrt{5}}{2} && \text{\# simplify} \\
&\Rightarrow \quad c = \frac{1 + \sqrt{5}}{2} && \text{\# positive root} \\
&\Rightarrow \quad c = \phi && \text{\# } \phi := \tfrac{1+\sqrt{5}}{2}
\end{aligned}
$$

@deilann@tech.lgbt also notes that $c^2 - c - 1$ is (or at least should be :-)) immediately identifiable as the golden ratio's quadratic form and ϕ's minimal polynomial which has ϕ and the negative inverse of ϕ as roots and so is "not sure solving it in full is truly necessary once you've gotten there".

LaTeX Source

`https://www.overleaf.com/read/mkjdjwtmnzjd`

References

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[3] Wikipedia Contributors. Quadratic Formula — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Quadratic_formula&oldid=1153992062`, 2023. [Online; accessed 11-May-2023].

[4] Wolfram MathWorld. Fundamental Theory of Calculus. `https://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html`, 2023. [Online; accessed 8-May-2023].

[5] Wolfram MathWorld. Inverse Hyperbolic Sine. `https://mathworld.wolfram.com/InverseHyperbolicSine.html`, 2023. [Online; accessed 8-May-2023].

Appendix A

This was my first attempt at proving Equation (2):

Equation (1) holds for a particular choice of the upper endpoint of the integral in Equation (2). In particular, Equation (1) holds when the upper endpoint of the integral equals $\frac{1}{2}$. More specifically:

$$
\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x^2+1}}\, dx &= \ln\left(x+\sqrt{x^2+1}\right)\Big|_0^{\frac{1}{2}} && \text{\# Equation (2) and the FToC} \\
&= \ln\left(\tfrac{1}{2}+\sqrt{\left(\tfrac{1}{2}\right)^2+1}\right)-\ln\left(0+\sqrt{0^2+1}\right) && \# \ f(x)\Big|_a^b \coloneqq f(b)-f(a) \\
&= \ln\left(\tfrac{1}{2}+\sqrt{\tfrac{5}{4}}\right)-\ln\left(0+\sqrt{0^2+1}\right) && \# \ \sqrt{\left(\tfrac{1}{2}\right)^2+1}=\sqrt{\tfrac{5}{4}} \\
&= \ln\left(\tfrac{1}{2}+\sqrt{\tfrac{5}{4}}\right)-\ln 1 && \# \ 0+\sqrt{0^2+1}=1 \\
&= \ln\left(\tfrac{1}{2}+\sqrt{\tfrac{5}{4}}\right) && \# \ \ln 1 = 0 \\
&= \ln\left(\tfrac{1+\sqrt{5}}{2}\right) && \# \ \tfrac{1}{2}+\sqrt{\tfrac{5}{4}}=\tfrac{1+\sqrt{5}}{2} \\
&= \ln\phi && \# \ \phi \coloneqq \tfrac{1+\sqrt{5}}{2}
\end{aligned}
$$

$\blacksquare$