Is i^i a real number and if so, what does it equal?

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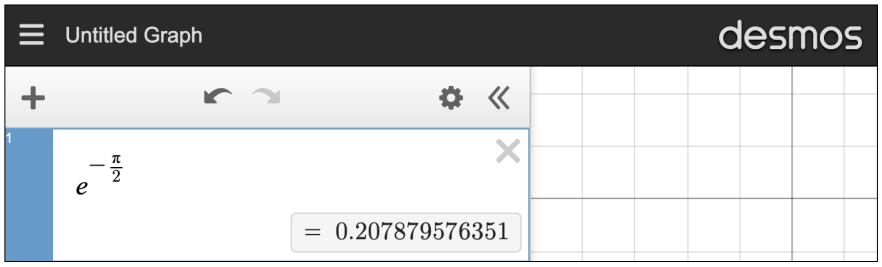
To answer this question, consider Euler's formula [1, 2] evaluated at $\frac{\pi}{2}$:

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x && \text{\# Euler's formula} \\
\Rightarrow\ e^{i\frac{\pi}{2}} &= \cos\tfrac{\pi}{2} + i \sin\tfrac{\pi}{2} && \text{\# set } x = \tfrac{\pi}{2} \\
\Rightarrow\ e^{i\frac{\pi}{2}} &= 0 + i \cdot 1 && \text{\# } \cos\tfrac{\pi}{2} = 0 \text{ and } \sin\tfrac{\pi}{2} = 1 \\
\Rightarrow\ e^{i\frac{\pi}{2}} &= i && \text{\# simplify} \\
\Rightarrow\ (e^{i\frac{\pi}{2}})^i &= i^i && \text{\# raise both sides to } i \\
\Rightarrow\ e^{\frac{i^2\pi}{2}} &= i^i && \text{\# } (x^m)^n = x^{mn} \\
\Rightarrow\ e^{-\frac{\pi}{2}} &= i^i && \text{\# } i^2 = -1 \\
\Rightarrow\ e^{-\frac{\pi}{2}} \in \mathbb{R} &\Rightarrow i^i \in \mathbb{R} && \text{\# } i^i \text{ is a real number}
\end{aligned}$$

Ok, $i^i \in \mathbb{R}$, but what does i^i equal? Well, we saw that $e^{-\frac{\pi}{2}} = i^i$. So

$$i^i = e^{-\frac{\pi}{2}} \approx 0.20788$$



References

[1] David Meyer. A Few Notes on Euler's Formula and Euler's Identity. https://davidmeyer.github.io/qc/euler.pdf, 2021. [See https://davidmeyer.github.io/qc/].

[2] Wikipedia contributors. Euler's Formula — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Euler%27s_formula&oldid=866429907, 2018. [Online; accessed 11-November-2018].