What does the series $S = \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n$ converge to?

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1 First up: does this series converge?

Here we'll use the ratio test for convergence [1] and so we want to think of S as

$$S = \sum_{n=1}^{\infty} a_n \tag{1}$$

where $a_n = \left(\frac{a}{b}\right)^n$.

The usual form of the ratio test uses the limit $L = \lim\limits_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. The ratio test tells us that

1. If $L < 1$ then the series converges absolutely.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ (or the limit doesn't exist) then the test is inconclusive.

To apply the ratio test we want to compute the following limit:

$$L = \lim_{n \to \infty} \left| \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} \right|$$

Since $\lim\limits_{n \to \infty} c = c$ for constant c and since $\dfrac{a}{b}$ is a constant with respect to n we see that the limit L is

$$L = \lim_{n \to \infty} \left| \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} \right| = \lim_{n \to \infty} \left| \frac{a}{b} \right| = \frac{a}{b}$$

If $a < b$ then $\dfrac{a}{b} < 1$ and so by clause 1 of the ratio test we know that S converges absolutely.

2 Ok, S converges. What does it converge to?

Since we know that S converges absolutely when $a < b$, here is one way to think about the question:

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \left(\tfrac{a}{b}\right)^n && \text{\# definiton of } S \text{ (Equation (1))} \\
&= \left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \cdots && \text{\# expand } S \\
&\Rightarrow \left(\tfrac{b}{a}\right) \cdot S = \left(\tfrac{b}{a}\right) \cdot \left[\left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \ \cdots\right] && \text{\# multiply both sides by } \left(\tfrac{b}{a}\right) \\
&\Rightarrow \left(\tfrac{b}{a}\right) \cdot S = 1 + \left[\left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \ \cdots\right] && \text{\# multiply through on right side} \\
&\Rightarrow \left(\tfrac{b}{a}\right) \cdot S = 1 + S && \text{\# definition of } S \\
&\Rightarrow \left(\tfrac{b}{a}\right) \cdot S - S = 1 && \text{\# subtract } S \text{ from both sides} \\
&\Rightarrow \left(\tfrac{b}{a}\right) \cdot S - \left(\tfrac{a}{a}\right) \cdot S = 1 && \text{\# multiply } S \text{ by } 1 = \tfrac{a}{a} \\
&\Rightarrow S \cdot \left[\tfrac{b}{a} - \tfrac{a}{a}\right] = 1 && \text{\# factor out } S \\
&\Rightarrow S \cdot \left[\tfrac{b-a}{a}\right] = 1 && \text{\# simplify} \\
&\Rightarrow S = \tfrac{a}{b-a} && \text{\# multiply both sides by } \tfrac{a}{b-a}
\end{aligned}
$$

So $S = \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n = \frac{a}{b-a}$, where $a, b \in \mathbb{N}$ and $a < b$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3-1} = \frac{1}{2}$.

Acknowledgements

LaTeX Source

References

[1] Wikipedia Contributors. Ratio Test — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Ratio_test&oldid=1075364794`, 2022. [Online; accessed 30-March-2022].