A Few Notes on Kepler Triangles

David Meyer

dmm613@gmail.com

Last Update: May 16, 2024

Initial Version: May 10, 2024

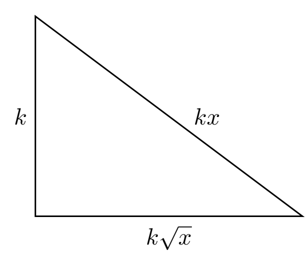
1 Introduction

A Kepler triangle is a right triangle whose sides are related by a geometric progression [2, 5] where the ratio of the progression is $\sqrt{\phi}$, the square root of the golden ratio ϕ [3]. The Kepler triangle has sides $1 : \sqrt{\phi} : \phi$. The triangle is named after the famous German astronomer and mathematician Johannes Kepler [4].

More generally, if the ratio of the geometric progression is $\sqrt{x}$, then the sides of the triangle are in the ratio $1 : \sqrt{x} : x$. From this we know (by the Pythagorean theorem [1]) that $1^2 + (\sqrt{x})^2 = x^2$. Simplifying we get $1 + x = x^2$, or in a more standard form $x^2 - x - 1 = 0$. This polynomial turns out to be the minimal characteristic polynomial of the golden ratio ϕ, so we know that $x = \phi$.

Interestingly, a triangle with sides k, $k\sqrt{x}$, and kx is also a Kepler triangle. This is because the ratio of the sides of this triangle is $k : k\sqrt{x} : kx$, and if we divide each side by k we see that the ratio of this triangle's sides is $1 : \sqrt{x} : x$. $\blacksquare$

We can also see this by considering following triangle:



By the Pythagorean theorem we know that

$$\begin{array}{rcll}
(kx)^2 & = & (k\sqrt{x})^2 + k^2 & \text{\# by the Pythagorean theorem} \\
& \Rightarrow & (kx)^2 - (k\sqrt{x})^2 - k^2 = 0 & \text{\# subtract } \left((k\sqrt{x})^2 + k^2\right) \text{ from both sides} \\
& \Rightarrow & k^2x^2 - k^2x - k^2 = 0 & \text{\# square terms} \\
& \Rightarrow & k^2(x^2 - x - 1) = 0 & \text{\# factor out } k^2 \\
& \Rightarrow & x^2 - x - 1 = 0 & \text{\# divide both sides by } k^2
\end{array}$$

And again, we know that $x^2 - x - 1$ is ϕ's minimal characteristic polynomial and so $x = \phi = \frac{1+\sqrt{5}}{2}$ and the ratio of the sides of this triangle is $1 : \sqrt{\phi} : \phi$. $\blacksquare$

2 Conclusions

Acknowledgements

LaTeX Source

https://www.overleaf.com/read/btgfkkdzktbt#adfb58

References

[1] Wikipedia Contributors. Pythagorean trigonometric identity — Wikipedia, the free encyclopedia, 2018. [Online; accessed 12-November-2018].

[2] Wikipedia Contributors. Geometric Series — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Geometric_series&oldid=1097138657, 2022. [Online; accessed 11-July-2022].

[3] Wikipedia Contributors. Golden Ratio — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Golden_ratio&oldid=1150269490, 2023. [Online; accessed 7-May-2023].

[4] Wikipedia contributors. Johannes Kepler — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Johannes_Kepler&oldid=1222947599, 2024. [Online; accessed 10-May-2024].

[5] Wikipedia contributors. Kepler triangle — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Kepler_triangle&oldid=1193796348, 2024. [Online; accessed 10-May-2024].