A Bit on Ramanujan and Nested Radicals

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1 Introduction

In 1911, the Indian mathematical genius Srinivasa Ramanujan posed the following problem in the Journal of the Indian Mathematical Society [1]: What does the nested radical shown in Equation (1) equal?

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}} \tag{1}$$

Having not received an answer for a few months, Ramanujan solved it himself. In these notes we look at Ramanujan's solution to a more general form of this problem.

Aside: I did a special case of this problem in "What is hiding inside the number 3?" [2], which is reproduced in Appendix A.

2 Ramanujan's Approach

What Ramanujan spotted was that for any non-negative integer x we have

$$\begin{aligned}
x+1 &= \sqrt{(x+1)^2} && \# \; x+1 = \sqrt{(x+1)^2} \\
&= \sqrt{1+2x+x^2} && \# \; (x+1)^2 = x^2+2x+1 \\
&= \sqrt{1+x(x+2)} && \# \; x^2+2x+1 = 1+x(x+2)
\end{aligned}$$

Next, notice that rewriting $(x+2)$ as $(x+1)+1$ gives us

$$\begin{aligned}
x+1 &= \sqrt{1+x((x+1)+1)} && \# \; (x+2) = (x+1)+1 \\
&= \sqrt{1+x\sqrt{((x+1)+1)^2}} && \# \; (x+1)+1 = \sqrt{((x+1)+1)^2} \\
&= \sqrt{1+x\sqrt{1+2(x+1)+(x+1)^2}} && \# \; ((x+1)+1)^2 = 1+\underbrace{2(x+1)+(x+1)^2}_{x^2+4x+3} \\
&= \sqrt{1+x\sqrt{1+(x+1)(x+3)}} && \# \; x^2+4x+3 = (x+1)(x+3)
\end{aligned}$$

Continuing, we can rewrite $(x+3)$ as $(x+2)+1$ and so

$$
\begin{aligned}
x+1 &= \sqrt{1+x\sqrt{1+(x+1)((x+2)+1)}} && \# \; (x+3) = (x+2)+1 \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{((x+2)+1)^2}}} && \# \; (x+2)+1 = \sqrt{((x+2)+1)^2} \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+2(x+2)+(x+2)^2}}} && \# \; ((x+2)+1)^2 = 1+2(x+2)+(x+2)^2 \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+x^2+6x+8}}} && \# \; 2(x+2)+(x+2)^2 = x^2+6x+8 \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}} && \# \; x^2+6x+8 = (x+2)(x+4) \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{((x+3)+1)^2}}}} && \# \; (x+4) = (x+3)+1 = \sqrt{((x+3)+1)^2} \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+x^2+8x+15}}}} && \# \; ((x+3)+1)^2 = 1+x^2+8x+15 \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)(x+5)}}}} && \# \; x^2+8x+15 = (x+3)(x+5) \\
&= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)\sqrt{((x+4)+1)^2}}}}} && \# \; x+5 = (x+4)+1 = \sqrt{((x+4)+1)^2}
\end{aligned}
$$

Now we can see that the general form of the expression is:

$$
x+1 = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)\sqrt{1+(x+4)\sqrt{1+(x+5)\sqrt{1+\ldots}}}}}}} \tag{2}
$$

Setting $x=2$ in Equation (2) we get

$$
3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+\cdots}}}}}}
$$

which is the result we saw in [2]. But now we have the general formula so we can plug in any positive integer, say $x=124$ for example:

$$
125 = \sqrt{1+124\sqrt{1+125\sqrt{1+126\sqrt{1+127\sqrt{1+\cdots}}}}}
$$

Acknowledgements

LaTeX Source

https://www.overleaf.com/read/qwhvvhrzrgct

References

[1] B.Sury. Ramanujan's route to roots of roots. https://www.isibang.ac.in/~sury/ramanujanday.pdf. [Online; accessed 14-June-2022].

[2] David Meyer. What is hiding inside the number 3? https://davidmeyer.github.io/qc/three.pdf, 2022. [Online; accessed 14-June-2022].

Appendix A: What is hiding inside the number 3?

Well, as we saw in [2]:

$$
\begin{aligned}
3 &= \sqrt{9} && \# \; 3 = \sqrt{9} \\
&= \sqrt{1+8} && \# \; 9 = 1+8 \\
&= \sqrt{1+2*4} && \# \; 8 = 2*4 \\
&= \sqrt{1+2\sqrt{16}} && \# \; 4 = \sqrt{16} \\
&= \sqrt{1+2\sqrt{1+15}} && \# \; 16 = 1+15 \\
&= \sqrt{1+2\sqrt{1+3*5}} && \# \; 15 = 3*5 \\
&= \sqrt{1+2\sqrt{1+3\sqrt{25}}} && \# \; 5 = \sqrt{25} \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+24}}} && \# \; 25 = 1+24 \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4*6}}} && \# \; 24 = 4*6 \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} && \# \; 6 = \sqrt{36} \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+35}}}} && \# \; 36 = 1+35 \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5*7}}}} && \# \; 35 = 5*7 \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{49}}}}} && \# \; 7 = \sqrt{49} \\
&= \cdots && \# \; 49 = 1+48, \; 48 = 6*8, \; 8 = \sqrt{64}, \; \ldots
\end{aligned}
$$

and so apparently $3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+\cdots}}}}}}$