$\pi^\pi = ?$

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Can we find an expression for π^π? The first thing we might do is to consider what we know about x^x. We do know that $a^x = e^{x \ln a}$ for positive a, since

$$\begin{aligned}
y &= a^x && \text{\# define } y \\
&\Rightarrow \ln y = \ln a^x && \text{\# take the log of both sides} \\
&\Rightarrow \ln y = x \ln a && \text{\# power rule for logarithms} \\
&\Rightarrow e^{\ln y} = e^{x \ln a} && \text{\# exponentiate both sides} \\
&\Rightarrow y = e^{x \ln a} && \text{\# } e^{\ln y} = y \\
&\Rightarrow a^x = e^{x \ln a} && \text{\# } y = a^x
\end{aligned}$$

We can use the same reasoning to show that $x^x = e^{x \ln x}$ for $x > 0$. Then setting $x = \pi$ we get

$$\pi^\pi = e^{\pi \ln \pi} \tag{1}$$

All good, but what is $e^{\pi \ln \pi}$? We can use a Maclaurin series to evaluate this expression as follows:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && \text{\# Maclaurin series for } e^x \\
&\Rightarrow e^{\pi \ln \pi} = \sum_{n=0}^{\infty} \frac{(\pi \ln \pi)^n}{n!} && \text{\# set } x = \pi \ln \pi \\
&\Rightarrow e^{\pi \ln \pi} = \sum_{n=0}^{\infty} \frac{\pi^n \ln^n \pi}{n!} && \text{\# simplify} \\
&\Rightarrow \pi^\pi = \sum_{n=0}^{\infty} \frac{\pi^n \ln^n \pi}{n!} && \text{\# } e^{\pi \ln \pi} = \pi^\pi \text{ (Equation (1))}
\end{aligned}$$

So we get the cool result that

$$\pi^\pi = \sum_{n=0}^{\infty} \frac{\pi^n \ln^n \pi}{n!}$$

Next question: is π^π rational or irrational?