

# A Few Notes on the Principle of Least Action and Newton's Second Law of Motion

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## 1 Introduction

The principle of least action, which is also known as the stationary-action principle, is a variational principle that, when applied to the action of a mechanical system, amazingly yields the equations of motion for that system. The principle states that the trajectories (i.e. the solutions of the equations of motion) are stationary points of the system's action functional [3]. One of the many amazing aspects of the principle is that it can be used to derive Newton's Second Law, Lagrangian and Hamiltonian equations of motion, and even general relativity (see Einstein–Hilbert action [5]).

These notes explore a bit of the connection between the Principle of Least Action [14] and Newton's Second Law [11].

## 2 Finding the Minimum of a Function

The story starts with how we use calculus to find the minimum of a function. In particular, suppose we have some function  $f(x)$ . We know that if we move a small distance from  $x$ , call it  $\epsilon$ , around  $f$ 's minimum<sup>1</sup> then  $f(x) = f(x + \epsilon)$  and so  $f(x + \epsilon) - f(x) = 0$ . For example consider  $f(x) = x^2 + 1$ . This case is shown in Figure 1, where we can see that  $f(0) = f(0 + \epsilon) = 1$  and  $\frac{d}{dx}f(0) = \frac{d}{dx}f(0 + \epsilon) = 0$ .

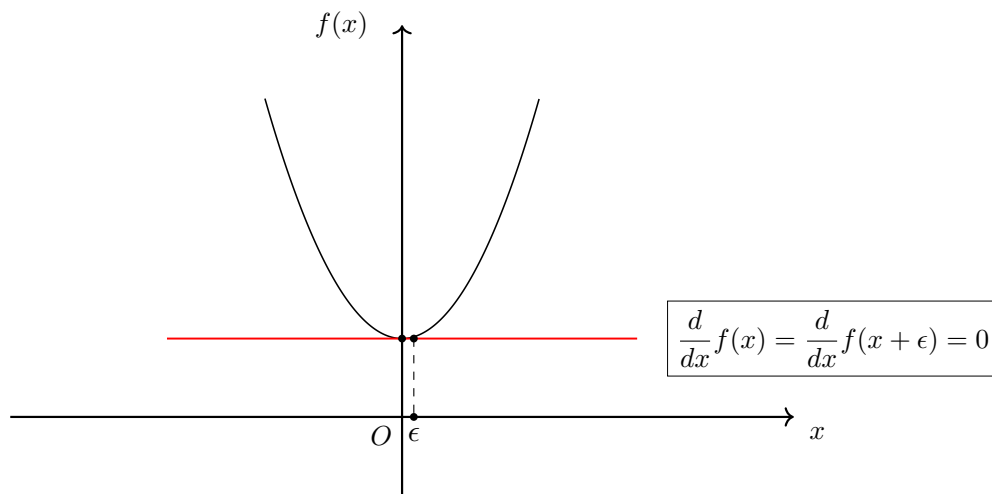


Figure 1:  $f(x) = x^2 + 1$

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<sup>1</sup>  $\frac{df}{dx} = 0$  at  $f$ 's minimum, also known as a *stationary point* of  $f$ .

We also know that we can evaluate  $f(x + \epsilon)$  using a Taylor series [15]. In particular

$$f(x + \epsilon) = f(x) + \epsilon \frac{df}{dx} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} + \frac{\epsilon^3}{3!} \frac{d^3 f}{dx^3} + \dots \quad \# \text{ definition of the Taylor series for } f(x + \epsilon)$$

Now if we define  $\Delta f = f(x + \epsilon) - f(x)$  we see that

$$\begin{aligned} \Delta f &= f(x + \epsilon) - f(x) && \# \text{ definition of } \Delta f \\ &= \left[ f(x) + \epsilon \frac{df}{dx} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} + \frac{\epsilon^3}{3!} \frac{d^3 f}{dx^3} + \dots \right] - f(x) && \# \text{ substitute Taylor series for } f(x + \epsilon) \\ &= \epsilon \frac{df}{dx} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} + \text{higher order terms} && \# f(x) \text{ cancels} \\ &= 0 + \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} + \dots && \# \frac{df}{dx} = 0 \text{ at } f\text{'s minimum} \\ &= \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} + \dots && \# \text{ simplify} \\ &\approx \frac{\epsilon^2}{2!} \frac{d^2 f}{dx^2} && \# \epsilon \text{ is close to zero so we can ignore higher order terms} \end{aligned}$$

So we can say that  $\Delta f \sim \epsilon^2$ , that is, the difference is a second order effect in  $\epsilon$ . Said from the perspective of first-order variation we can say that the first-order variation in the value of  $f$  is constant in some neighborhood around  $f$ 's stationary point (that is, where the gradient of  $f$  equals zero).

So in summary we know that for normal functions  $f$  at the minimum

$$\Delta f = f(x + \epsilon) - f(x) = 0 \quad (1)$$

at least to the first order in  $\epsilon$ . The constraint specified by Equation 1 will become important when we study  $\delta S$ .

Another way to see this is that if we only consider the first order terms of the Taylor series we see that

$$\begin{aligned} \Delta f &= f(x + \epsilon) - f(x) && \# \text{ definition of } \Delta f \text{ (Equation 1)} \\ &= f(x) + \epsilon \frac{df}{dx} - f(x) && \# \text{ consider only 1}^{\text{st}} \text{ order effects so only use the first 2 terms of the Taylor series for } f(x + \epsilon) \\ &= \epsilon \frac{df}{dx} && \# f(x) \text{ cancels} \\ &= 0 && \# \frac{df}{dx} = 0 \text{ at the extremums [10]} \end{aligned}$$

So we can see that  $\Delta f$  must equal zero, at least to the first order in  $\epsilon$ .

An important (or maybe, the most important) difference between the approach taken by the Principle of Least Action and Newton's Second Law is that the Principle of Least Action looks at the value of an entire *path*, where as Newton's Second Law looks at points along the path. That is, the Newtonian approach is local while the Principle of Least Action approach is global. More specifically, in Newton's case we are given initial conditions (the initial position and the initial velocity) and then solve the differential equation ( $F = ma$ ) for  $x(t)$ . Put another way, in Newtonian mechanics we solve  $-\frac{\partial V}{\partial x} = m \frac{d^2 x}{dt^2}$  to find  $x(t)$ . As we will see in Section 3, Lagrangian mechanics takes a different approach: we find an object's trajectory  $x(t)$  from its beginning and ending positions (the boundary values). Amazingly these two approaches turn out to be equivalent.

### 3 The Principle of Least Action

The action, denoted  $\mathcal{S}$ , is defined to be the sum of the difference between a particle's kinetic energy [8] and its potential energy [12] at every point along its path.  $\mathcal{S}$  is just a number that is associated with the trajectory (path). The Principle of Least Action states that the path for which the sum of these differences (that is,  $\mathcal{S}$ ) is minimum is the actual path the particle will take. To show that this is the case all of the proofs that I have seen rely on the fact that the Principle of Least Action and Newton's Second Law ( $F = ma$ ) are equivalent. These notes show this result below.

To give a bit more formal definition of the Principle of Least Action:

**Definition 3.1. The Principle of Least (Stationary) Action:** The path of a particle is the one that yields a stationary value of the action. A stationary point of some function is a point where its gradient is equal to zero. In other words, the first-order variation in the value of the function is constant in some neighborhood around the stationary point. As we will see below, this condition can be expressed as  $\delta\mathcal{S} = 0$ .

Consider the setup shown in Figure 2 and imagine that the actual path is represented by the function  $x(t)$ . That is,  $x(t)$  represents the path of a particle that starts out at position  $x_1$  at time  $t_1$  and ends up at  $x_2$  at time  $t_2$ . Then what we want to do is to look at another path that is some small deviation away from  $x(t)$  (we know from Equation 1 that this will be close to  $x(t)$ ). We hope that if we look at the change in the action of the path with the small deviation that change should be close to zero as well.

As we can see from Figure 2 we call the small deviation  $\eta(t)$  and we call the deviated path  $x(t) + \eta(t)$ .

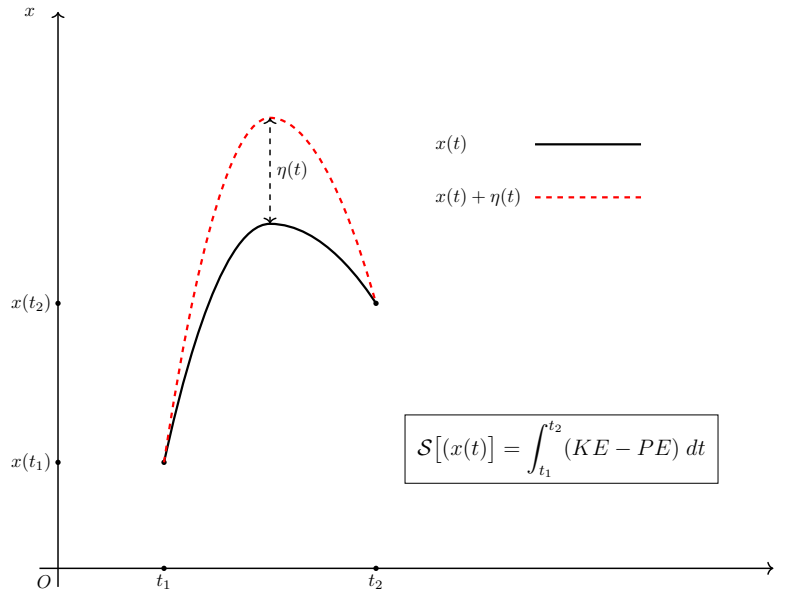


Figure 2: Principle of Least Action Setup

Now, consider a particle moving in one dimension in a uniform gravitational field  $g$  with potential energy  $V(x)^2$ . In this case  $V(x) = mgx$ . So

$$\begin{aligned} \mathcal{S}[x(t)] &= \int_{t_1}^{t_2} (KE - PE) dt && \# \text{ definition of } \mathcal{S}[x(t)] \\ &= \int_{t_1}^{t_2} \left[ \frac{1}{2}m \left( \frac{dx}{dt} \right)^2 - V(x) \right] dt && \# KE = \frac{1}{2}m \left( \frac{dx}{dt} \right)^2, PE = V(x) \end{aligned}$$

<sup>2</sup> $V(x)$  is sometimes called  $U(x)$ , and is also informally referred to as  $PE$ .

Note that  $\mathcal{S}$  is a *functional* [4] so we use square brackets (e.g.  $\mathcal{S}[x(t)]$ ) instead of parenthesis to distinguish the functional from the function (e.g  $S(x)$ ). Now, abbreviating  $x(t)$  by  $x$  we see that

$$\mathcal{S}[x] = \int_{t_1}^{t_2} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) \right] dt \quad (2)$$

Now define  $\delta\mathcal{S} = \mathcal{S}[x + \eta] - \mathcal{S}[x]$ . We know that the path taken by the system,  $x(t)$ , has a stationary action ( $\delta\mathcal{S} = 0$ ) under small changes to the configuration of the system. Said another way, we know that  $\delta\mathcal{S} = 0$  to the first order in  $\eta$  (here again  $x$  is shorthand for  $x(t)$  and  $\eta$  is shorthand for  $\eta(t)$ ). So now we can expand  $\mathcal{S}[x + \eta]$  as follows:

$$\begin{aligned} \mathcal{S}[x + \eta] &= \int_{t_1}^{t_2} \left[ \frac{1}{2} m \left( \frac{d}{dt}(x + \eta) \right)^2 - V(x + \eta) \right] dt && \# \text{ substitute } x + \eta \text{ for } x \\ &= \int_{t_1}^{t_2} \left[ \frac{1}{2} m \left( \frac{dx}{dt} + \frac{d\eta}{dt} \right)^2 - V(x + \eta) \right] dt && \# \text{ derivative is a linear operator} \end{aligned} \quad (3)$$

Next notice that

$$\begin{aligned} \left( \frac{dx}{dt} + \frac{d\eta}{dt} \right)^2 &= \left( \frac{dx}{dt} \right)^2 + \left( \frac{d\eta}{dt} \right)^2 + 2 \left( \frac{dx}{dt} \right) \left( \frac{d\eta}{dt} \right) && \# \text{ multiply out} \\ &= \left( \frac{dx}{dt} \right)^2 + 2 \left( \frac{dx}{dt} \right) \left( \frac{d\eta}{dt} \right) && \# \left( \frac{d\eta}{dt} \right)^2 \approx 0 \text{ (Section 2)} \end{aligned}$$

Next want to expand  $V(x + \eta)$  using a Taylor series:

$$V(x + \eta) = V(x) + \eta \frac{dV}{dx} + \frac{\eta^2}{2!} \frac{d^2V}{dx^2} + \dots \quad \# \text{ expand Taylor series}$$

So now if we substitute these last two expressions into Equation 3 and group terms we see that

$$\mathcal{S}[x + \eta] = \int_{t_1}^{t_2} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) + \frac{1}{2} m \left( 2 \frac{dx}{dt} \frac{d\eta}{dt} \right) - \eta \frac{dV}{dx} - \text{higher order terms} \right] dt$$

We also have the constraint that

$$\delta\mathcal{S} = \mathcal{S}[x + \eta] - \mathcal{S}[x] = 0$$

and we saw earlier (Equation 2) that

$$\mathcal{S}[x] = \int_{t_1}^{t_2} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) \right] dt$$

Putting the last three equations together (and ignoring higher order terms in the Taylor series for  $\mathcal{S}(x + \eta)$ ) we find that

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left[ m \left( \frac{dx}{dt} \frac{d\eta}{dt} \right) - \eta \frac{dV}{dx} \right] dt = 0$$

What we would like is to have this integral be zero regardless of the value of  $\eta$ . So how can we "factor out"  $\eta$ ? The first thing to notice is that by the product rule [13] we have

$$\begin{aligned} \frac{d}{dx} \left( \frac{dx}{dt} \cdot \eta \right) &= \frac{d^2x}{dt^2} \cdot \eta + \frac{dx}{dt} \cdot \frac{d\eta}{dt} && \# \text{ product rule: } \frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \\ \Rightarrow \frac{dx}{dt} \frac{d\eta}{dt} &= \frac{d}{dx} \left( \frac{dx}{dt} \eta \right) - \frac{d^2x}{dt^2} \eta && \# \text{ solve for } \frac{dx}{dt} \frac{d\eta}{dt} \\ \Rightarrow \int_{t_1}^{t_2} \frac{dx}{dt} \frac{d\eta}{dt} &= \int_{t_1}^{t_2} \frac{d}{dx} \left( \frac{dx}{dt} \eta \right) dt - \int_{t_1}^{t_2} \frac{d^2x}{dt^2} \eta dt && \# \text{ integrate both sides} \\ \Rightarrow \int_{t_1}^{t_2} \frac{dx}{dt} \frac{d\eta}{dt} &= \frac{dx}{dt} \eta \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2x}{dt^2} \eta dt && \# \text{ integral \& derivative in } 1^{st} \text{ term cancel} \\ \Rightarrow \int_{t_1}^{t_2} \frac{dx}{dt} \frac{d\eta}{dt} &= - \int_{t_1}^{t_2} \frac{d^2x}{dt^2} \eta dt && \# \eta(t_1) = \eta(t_2) = 0 \text{ (by definition)} \end{aligned}$$

So now we know that

$$\int_{t_1}^{t_2} \frac{dx}{dt} \frac{d\eta}{dt} = - \int_{t_1}^{t_2} \frac{d^2x}{dt^2} \eta dt \quad (4)$$

and we saw earlier that

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left[ m \left( \frac{dx}{dt} \frac{d\eta}{dt} \right) - \frac{dV}{dx} \eta \right] dt = 0 \quad (5)$$

so

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_1}^{t_2} \left[ -m \frac{d^2x}{dt^2} \eta - \frac{dV}{dx} \eta \right] dt && \# \text{ substitute result from Equation 4 into Equation 5} \\ &= \int_{t_1}^{t_2} \left[ -m \frac{d^2x}{dt^2} - \frac{dV}{dx} \right] \eta dt && \# \text{ factor out } \eta \end{aligned}$$

Since  $\delta\mathcal{S}$  must equal zero (Equation 5) and we know  $\eta(t)$  is not the constant function 0 so it must be that

$$-m \frac{d^2x}{dt^2} - \frac{dV}{dx} = 0$$

But now notice that

$$-\frac{dV}{dx} = m \frac{d^2x}{dt^2}$$

and since  $-\frac{dV}{dx} = F$  and  $\frac{d^2x}{dt^2} = a$  we amazingly have arrived at Newton's Second Law [11]:

$$F = ma$$

Finally, in the general case we write the action  $\mathcal{S}$  as

$$\mathcal{S} \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

where  $L$  is called the *Lagrangian* (which we will discuss in more detail in Section 4) and  $\dot{x}$  is shorthand for  $\frac{dx}{dt}$ . One other detail: the variable  $x$  here can refer to any number of coordinates, the so-called *generalized coordinates*, the  $i^{\text{th}}$  of which is conventionally denoted  $q_i$ .

## 4 The Euler-Lagrange Equation

Consider the following seemingly counter-intuitive combination of the kinetic and potential energies ( $T$  and  $V$  respectively):

$$L := T - V \tag{6}$$

$L$  here is called the Lagrangian. The Lagrangian is essentially the action per unit time and as we saw above, in classical mechanics this particular Lagrangian ( $L = T - V$ ) is precisely the Lagrangian that generates Newton's second law ( $F = ma$ ). Next we'll briefly look at an application of the Lagrangian which involves a simple harmonic oscillator, specifically a mass on the end of a spring. We will study this problem in a bit more detail in Section 5.

Here the kinetic energy  $T$  is  $\frac{1}{2}m\dot{x}^2$  and the spring potential energy  $V$ , given by Hooke's Law [7], write the Lagrangian as follows:

$$L = KE - PE = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

We need one more piece of the puzzle before we can solve for  $x$  as a function of time, the *Euler-Lagrange* equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \tag{7}$$

Essentially the Euler-Lagrange equation is a constraint that the Lagrangian must satisfy in order for the principle of stationary action to be true. It is essentially what generates the equations of motion of a system given a specific Lagrangian, just as Newton's second law does for a given set of forces.

To find the position of the mass  $m$  as a function of time (call it  $x(t)$ ) we need to solve the Euler-Lagrange equation for this value of  $L$ . Recall that the Euler-Lagrange equation is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

So first we want to compute  $\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right]$ . Since differentiation is a linear operation [9] and we know that  $\frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 \right] = 0$  we can see that

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right] = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 \right] - \frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = 0 - \frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = -\frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = -\frac{1}{2}2kx = -kx$$

Next, we want to find  $\frac{\partial L}{\partial \dot{x}}$  and then take its derivative with respect to  $t$  (time). Here we know that  $\frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} k x^2 \right] = 0$  and so

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right] = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} m \dot{x}^2 \right] - \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} k x^2 \right] = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} m \dot{x}^2 \right] - 0 = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2} m \dot{x}^2 \right] = \frac{1}{2} 2 m \dot{x} = m \dot{x}$$

Now we can see that  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{d}{dt} (m \dot{x}) = m \ddot{x}$ . Plugging these values back into the Euler-Lagrange equation we find that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m \ddot{x} - (-kx) = m \ddot{x} + kx \Rightarrow \ddot{x} + \frac{k}{m} x = 0$$

Here we recognize that  $\ddot{x} + \frac{k}{m} x = 0$  is a differential equation that has a solution in sine or cosine (doesn't matter which) where  $\frac{k}{m}$  represents the angular frequency. If we let  $\omega_0 = \sqrt{\frac{k}{m}}$  then  $\omega_0^2 = \frac{k}{m}$  and so  $\ddot{x} + \omega_0^2 x = 0$ . But this is a differential equation that has the general solution [1]

$$x(t) = A \cos(\omega_0 t - \phi)$$

where  $A$  is the amplitude,  $\omega_0$  is the angular frequency and  $\phi$  is phase. Note that  $A$  and  $\phi$  depend on initial conditions while  $\omega_0$  does not.

So in this simple system the position of the mass as a function of time is given by  $x(t) = A \cos(\omega_0 t - \phi)$  for some amplitude  $A$  and where the angular frequency is given by the spring constant ( $k$ ) divided by the mass of the object ( $m$ ).

## 5 A Bit More Detail on the Simple Harmonic Oscillator Problem

In Section 4 we briefly considered the dynamics of a mass on the end of a spring and introduced the Euler-Lagrange equation (Equation 7) as a method to solve the system. In this section we'll consider this problem in a bit more detail and use the Euler-Lagrange equation to solve for the position of the mass as a function of time.

To start, consider the simple harmonic oscillator [6] shown in Figure 3. Here the mass  $m$  is moving back and forth in one dimension on a frictionless surface with spring constant  $k$  and equilibrium point EP.

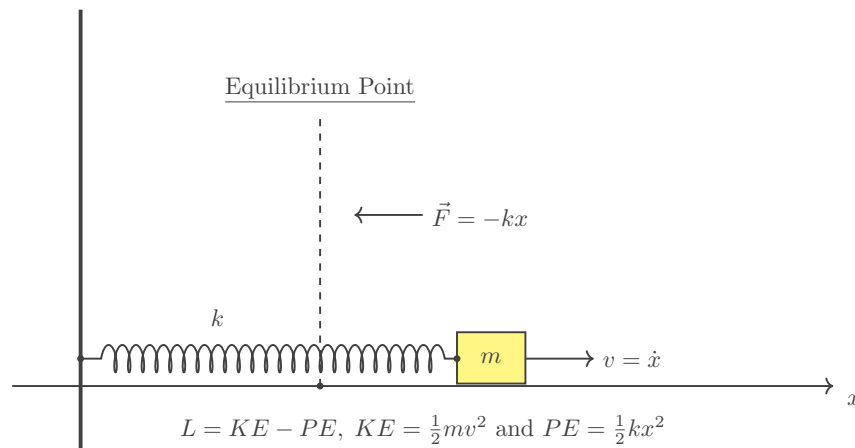


Figure 3: Simple Harmonic Oscillator with Spring Constant  $k$  and Mass  $m$

In this example we will be using generalized coordinates so we use  $v = \dot{x}$  and write the kinetic energy  $KE = \frac{1}{2}m\dot{x}^2$ . So we can write the Lagrangian as follows:

$$L = KE - PE = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

To find the position of the mass  $m$  as a function of time (call it  $x(t)$ ) we need to solve the Euler–Lagrange equation for this value of  $L$ . Recall that the Euler–Lagrange equation is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

So first we want to compute  $\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right]$ . Since differentiation is a linear operation [9] and we know that  $\frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 \right] = 0$  we can see that

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right] = \frac{\partial}{\partial x} \left[ \frac{1}{2}m\dot{x}^2 \right] - \frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = 0 - \frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = -\frac{\partial}{\partial x} \left[ \frac{1}{2}kx^2 \right] = -\frac{1}{2}2kx = -kx$$

Next, we want to find  $\frac{\partial L}{\partial \dot{x}}$  and then take its derivative with respect to  $t$  (time). Here we know that  $\frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}kx^2 \right] = 0$  and so

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right] = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}m\dot{x}^2 \right] - \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}kx^2 \right] = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}m\dot{x}^2 \right] - 0 = \frac{\partial}{\partial \dot{x}} \left[ \frac{1}{2}m\dot{x}^2 \right] = \frac{1}{2}2m\dot{x} = m\dot{x}$$

Now we can see that  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{d}{dt} (m\dot{x}) = m\ddot{x}$ . Plugging these values back into the Euler–Lagrange equation we find that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m\ddot{x} - (-kx) = m\ddot{x} + kx \Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

Here we recognize that  $\ddot{x} + \frac{k}{m}x = 0$  is a differential equation that has a solution in sine or cosine (doesn't matter which) where  $\frac{k}{m}$  represents the angular frequency. If we let  $\omega_0 = \sqrt{\frac{k}{m}}$  then  $\omega_0^2 = \frac{k}{m}$  and so  $\ddot{x} + \omega_0^2 x = 0$ . But this is a differential equation that has the general solution [1]

$$x(t) = A \cos(\omega_0 t - \phi)$$

where  $A$  is the amplitude,  $\omega_0$  is the angular frequency and  $\phi$  is phase. Note that  $A$  and  $\phi$  depend on initial conditions while  $\omega_0$  does not.



## 6 Conclusions

It's amazing that two such different approaches, Newtonian mechanics and Lagrangian mechanics, turn out to be equivalent. BTW, Kenneth Young has a nice video on the Principle of Least Action [2].

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