A Useful Mathematical Trick, Telescoping Series, and the Infinite Sum of the Reciprocals of the Triangular Numbers

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1 A Useful Mathematical Trick: $\left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) = \frac{1}{n} - \frac{1}{n+1}$

This useful bit of mathematical magic works when the denominators of the two fractions are of the form n and $n+1$, their numerators are both 1, and $n > 0$. These are the so-called pronic numbers [7]. As an example, for $n = 3$ we get

$$\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12} \text{ and } \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

So why does this work? Consider

$$\begin{aligned}
\left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) &= \frac{1}{n \cdot (n+1)} && \text{\# multiply through} \\
&= \frac{(n-n)+1}{n \cdot (n+1)} && \text{\# } (n-n) = 0 \text{ and } 0+1 = 1 \\
&= \frac{(n+1)-n}{n \cdot (n+1)} && \text{\# rearrange (+ is commutative and associative)} \\
&= \frac{(n+1)}{n \cdot (n+1)} - \frac{n}{n \cdot (n+1)} && \text{\# split into a difference} \\
&= \frac{1}{n} - \frac{n}{n \cdot (n+1)} && \text{\# cancel } (n+1) \text{ in the first term} \\
&= \frac{1}{n} - \frac{1}{n+1} && \text{\# cancel } n \text{ in the second term}
\end{aligned}$$

So now we can see why

$$\left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) = \frac{1}{n} - \frac{1}{n+1} \tag{1}$$

An example of the use of this trick in evaluating infinite series is Burkard Polster's (aka Mathologer) use of Equation (1) when describing how Euler's solution to the Basel problem works [3, 4].

2 Telescoping Series

What is a telescoping series? Briefly, a telescoping series is a sum that is characterized by partial sums (called telescoping sums) that contain pairs of consecutive terms which cancel each other, leaving only the first and final terms [8]. This cancellation of adjacent terms is whimsically referred to as ”collapsing the telescope”.

For example, consider the infinite series whose terms are described by Equation (1). This is perhaps the canonical example of a telescoping series, which can be seen as follows:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) && \text{\# Equation (1)}\\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right) && \text{\# definition of an infinite series [10]}\\
&= \lim_{N\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{N}-\frac{1}{N+1}\right)\right] && \text{\# expand the sum}\\
&= \lim_{N\to\infty}\left[1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{N}+\frac{1}{N}\right)-\frac{1}{N+1}\right] && \text{\# group terms}\\
&= \lim_{N\to\infty}\left[1-\frac{1}{N+1}\right] && \text{\# collapse the telescope}\\
&= \lim_{N\to\infty}[1]-\lim_{N\to\infty}\left[\frac{1}{N+1}\right] && \text{\# limit is a linear operator [2]}\\
&= 1-0 && \text{\#} \lim_{n\to\infty}[c]=c \text{ and } \lim_{n\to\infty}\left[\frac{1}{n+1}\right]=0\\
&= 1 && \text{\#} \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right)=1
\end{aligned}
$$

A bit more formally, a telescoping series is the sum of terms of the form $(a_j - a_{j+1})$. As a result of the structure of its terms a telescoping series has the following property:

$$
\begin{aligned}
\sum_{j=1}^{n}(a_j-a_{j+1}) &= (a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\cdots+(a_n-a_{n+1}) && \text{\# expand terms}\\
&= a_1+(a_2-a_2)+(a_3-a_3)+\cdots+(a_n-a_n)-a_{n+1} && \text{\# group like terms}\\
&= a_1-a_{n+1} && \text{\# simplify}
\end{aligned}
$$

3 Triangular Numbers

The example I want to explore here is the infinite sum of the reciprocals of the triangular numbers [9]. So first, what is a triangular number? The first six triangular numbers are depicted as triangles in Figure 1. We can see from the figure that a triangular number T_n is the sum of the integers from 1 to n.

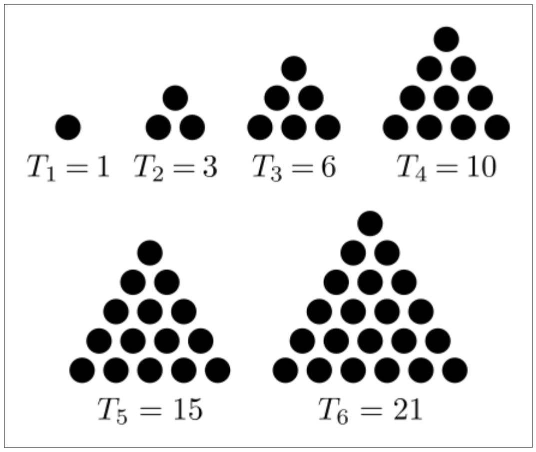

Figure 1: The First Six Triangular Numbers [9]

Ok, but what is the general form of T_n? Since T_n is the sum of the first n integers we know that[1]

$$T_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2} \tag{2}$$

As an aside, there is an interesting relationship between the triangular numbers and Pascal's Triangle [6]: if you start at the second row (counting from zero) and second column (again counting from zero) of Pascal's Triangle the corresponding diagonal contains the triangular numbers. This is shown in Figure 2. Figure 3 shows Pascal's Triangle with binomial coefficients (the triangular numbers are shown in blue).

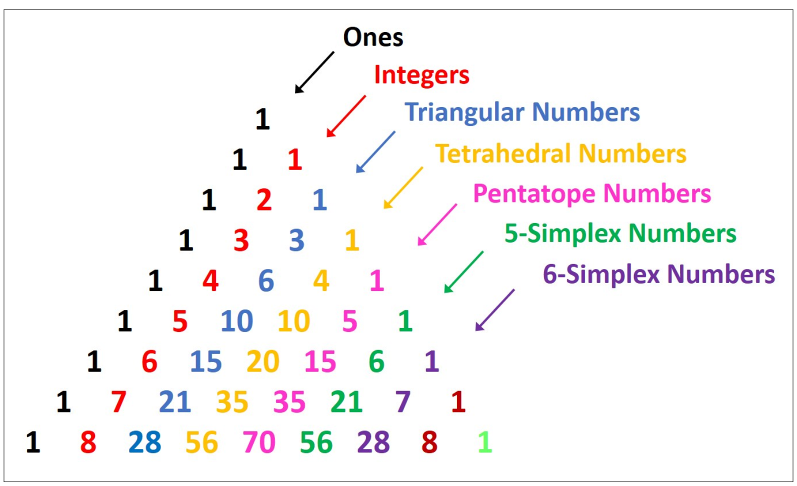

Figure 2: Number patterns in Pascal's Triangle [5]

[1]The great mathematician Carl Friedrich Gauss is said to have discovered the consecutive integer formula while still at primary school [1].

$$
\begin{array}{c}
\binom{0}{0} \\[6pt]
\binom{1}{0} \quad \binom{1}{1} \\[6pt]
\binom{2}{0} \quad \binom{2}{1} \quad \color{blue}{\binom{2}{2}} \\[6pt]
\binom{3}{0} \quad \binom{3}{1} \quad \color{blue}{\binom{3}{2}} \quad \binom{3}{3} \\[6pt]
\binom{4}{0} \quad \binom{4}{1} \quad \color{blue}{\binom{4}{2}} \quad \binom{4}{3} \quad \binom{4}{4} \\[6pt]
\binom{5}{0} \quad \binom{5}{1} \quad \color{blue}{\binom{5}{2}} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\[6pt]
\binom{6}{0} \quad \binom{6}{1} \quad \color{blue}{\binom{6}{2}} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\[6pt]
\binom{7}{0} \quad \binom{7}{1} \quad \color{blue}{\binom{7}{2}} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\[6pt]
\binom{8}{0} \quad \binom{8}{1} \quad \color{blue}{\binom{8}{2}} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}
\end{array}
$$

Figure 3: Pascal's Triangle with Binomial Coefficients

4 The Sum of the Reciprocals of the Triangular Numbers

All of this is good, but back to the question: what is the sum of the reciprocals of the triangular numbers?

Well, first, the sum $\displaystyle\sum_{n=1}^{N} \frac{1}{T_n}$ looks like

$$
\begin{aligned}
\sum_{n=1}^{N} \frac{1}{T_n} &= \sum_{n=1}^{N} \frac{1}{\frac{n(n+1)}{2}} && \text{\# definition of } T_n \text{ (Equation (2))} \\
&= \sum_{n=1}^{N} \frac{2}{n(n+1)} && \text{\# simplify} \\
&= 2\sum_{n=1}^{N} \left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) && \text{\# pull 2 out of the sum and group terms} \\
&= 2\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) && \text{\# Equation (1)}
\end{aligned}
$$

We saw in Section 2 that $\displaystyle\lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} \left[1 - \frac{1}{N+1}\right] = 1$ and we know from the above that the infinite sum of the reciprocals of the triangular numbers is twice this limit. So we know that

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2\left[\lim_{N\to\infty} \left[1 - \frac{1}{N+1}\right]\right] = 2 \cdot 1 = 2$$

5 Conclusions

Equation (1) turns out to be very useful, especially when evaluating infinite telescoping series. We also saw in Section 4 that interestingly, the infinite sum of the reciprocals of the triangular numbers equals two, that is,

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$$

As we saw in Section 2, the evaluation of this sum makes heavy use of a telescoping series described by Equation (1).

Acknowledgements

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