What is hiding inside the number 2?

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Consider

$$
\begin{aligned}
2 &= \sqrt{4} && \# \; 4 = 2^2 \\
&= \sqrt{2+2} && \# \; 4 = 2+2 \\
&= \sqrt{2+\sqrt{4}} && \# \; 4 = 2^2 \\
&= \sqrt{2+\sqrt{2+2}} && \# \; 4 = 2+2 \\
&= \sqrt{2+\sqrt{2+\sqrt{4}}} && \# \; 4 = 2^2 \\
&= \sqrt{2+\sqrt{2+\sqrt{2+2}}} && \# \; 4 = 2+2 \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4}}}} && \# \; 4 = 2^2 \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+2}}}} && \# \; 4 = 2+2 \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4}}}}} && \# \; 4 = 2^2 \\
&= \cdots
\end{aligned}
$$

and so apparently

$$2 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}$$

Acknowledgements

Thanks to Bruce Mah who pointed out that since $4 = 2+2 = 2 \cdot 2$ we can also write

$$2 = \sqrt{2 \cdot \sqrt{2 \cdot \sqrt{2 \cdot \sqrt{2 \cdot \sqrt{2 \cdot \cdots}}}}} \tag{1}$$

Amazing.

Dave Neary also pointed out that if you take the log of both sides of Equation (1) you get

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \tag{2}$$

Here we can notice that the right-hand side (RHS) of Equation (2) is a geometric series [1] with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. Since this geometric series converges to $\frac{a}{1-r}$, we see that

$$\begin{aligned}
1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots && \text{\# take the log of both sides of Equation (1)} \\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} && \text{\# the RHS is a geometric series that converges to } \tfrac{a}{1-r} \text{ with } a = r = \tfrac{1}{2} \\
&= \frac{\frac{1}{2}}{\frac{1}{2}} && \text{\# simplify} \\
&= 1 && \text{\# amazing}
\end{aligned}$$

References

[1] Wikipedia Contributors. Geometric Series — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Geometric_series&oldid=1097138657`, 2022. [Online; accessed 11-July-2022].