A Few Notes on Vector Calculus

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1 Vector Spaces

1.1 Definitions

Definition 1.1. A *field* is an algebraic structure[1] consisting of a non-empty set $\mathbb{K}$ equipped with two binary operations, $+$ (addition) and $\cdot$ (multiplication), satisfying the conditions[2]:

(A) $(\mathbb{K}, +)$ is an Abelian group with identity element 0 (called zero).

(M) $(\mathbb{K}\backslash\{0\}, \cdot)$ is an Abelian group with identity element 1.

(D) The distributive law $a(b + c) = ab + ac$ holds for all $a, b, c \in \mathbb{K}$.

Examples of important fields include

- $\mathbb{Q}$, the field of rational numbers
- $\mathbb{R}$, the field of real numbers
- $\mathbb{C}$, the field of complex numbers
- $\mathbb{Z}_p$, the field of integers mod p for prime p

Definition 1.2. A vector space V over a field $\mathbb{K}$ is an algebraic structure consisting of a non-empty set V equipped with a binary operation $+$ (vector addition) and a scalar multiplication operation $(a, v) \in \mathbb{K} \times V \mapsto av \in V$ such that the following rules hold:

(VA) $(V, +)$ is an Abelian group with identity element $\mathbf{0}$ (the zero vector).

(VM) Rules for scalar multiplication:

(VM0) Closure: For any $a \in \mathbb{K}$ and $v \in V$ there is a unique element $av \in V$.

(VM1) Distributivity$_1$: For any $a \in \mathbb{K}$ and $u, v \in V$ we have $a(u + v) = au + av$.

(VM2) Distributivity$_2$: For any $a, b \in \mathbb{K}$ and $v \in V$ we have $(a + b)v = av + bv$.

(VM3) Associativity: For any $a, b \in \mathbb{K}$ and $v \in V$ we have $(ab)v = a(bv)$.

(VM4) Identity: For any $v \in V$ we have $1v = v$ (where 1 is the identity element of $\mathbb{K}$).

Since vector spaces have two kinds of elements, namely elements of $\mathbb{K}$ and elements of V, we distinguish them by calling the elements of $\mathbb{K}$ scalars and the elements of V vectors.

A vector space over the field $\mathbb{R}$ is often called a real vector space while a vector space over $\mathbb{C}$ is called a complex vector space.

[1] See Appendix A for a brief review of a few important algebraic structures.
[2] See Appendix B for more on groups and fields.

1.2 Vectors

In this section we'll define the vector notation that we will use in these notes as well as a few important vector operations.

1.2.1 Notation

In these notes we will use boldface to represent a vector. Specifically, we will use $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ to represent a column or row vector in some n-dimensional space (usually $\mathbb{R}$ or $\mathbb{C}$). If $\mathbf{a}$ is a column vector then in matrix format

$$\mathbf{a} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{n\times 1}$$

Alternatively, if $\mathbf{a}$ is a row vector then in matrix format

$$\mathbf{a} = \underbrace{\begin{bmatrix} a_1 \; a_2 \; \ldots \; a_n \end{bmatrix}}_{1\times n}$$

The *transpose* of a vector $\mathbf{a}$, $\mathbf{a}^\intercal$, is defined as follows: If $\mathbf{a}$ is a column vector then

$$\mathbf{a}^\intercal = \begin{bmatrix} a_1 \; a_2 \; \ldots \; a_n \end{bmatrix}$$

Alternatively, if $\mathbf{a}$ is a row vector then

$$\mathbf{a}^\intercal = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Adding a "hat" to a vector denotes the unit vector. That is, for a vector $\mathbf{u}$, $\hat{\mathbf{u}}$ is defined to be

$$\hat{\mathbf{u}} := \frac{\mathbf{u}}{\|\mathbf{u}\|} \tag{1}$$

where $\|\mathbf{u}\|$ is the Euclidean Norm of the vector $\mathbf{u}$ (Definition 1.4).

In words: $\hat{\mathbf{u}}$ is a vector of unit length in the $\mathbf{u}$ direction.

Similarly, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in $\mathbb{R}^3$ in the x, y, and z directions respectively. Note that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the canonical basis vectors for $\mathbb{R}^3$ [4] and have column vector format

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

I have also seen $\mathbf{e}_i$ used to represent the i^{th} basis vector in $\mathbb{R}^n$. So $\mathbf{e}_i$ is a vector with a one in the i^{th} position and a zero in each of the other $n-1$ positions. In $\mathbb{R}^3$ this means that $\mathbf{e}_1 = \hat{\mathbf{i}}$, $\mathbf{e}_2 = \hat{\mathbf{j}}$, and $\mathbf{e}_3 = \hat{\mathbf{k}}$.

In general, the standard basis (which is sometimes called the computational basis) for the n-dimensional Euclidean space consists of the ordered set of n distinct vectors

$$\{\mathbf{e}_i : 1 \leq i \leq n\}$$

where $\mathbf{e}_i$ is the i^{th} basis vector, that is, it has a one in the i^{th} coordinate (position) and zeros everywhere else[3]. The $\mathbf{e}_i$ have column vector format

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \ \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Using these definitions we can define the parametric form of some curve C in $\mathbb{R}^3$, $\mathbf{r}(t)$, as follows:

$$\mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}$$

where $t \in [a, b]$ and $g(t)$, $h(t)$, and $k(t)$ are scalar functions[4].

Another common notation for vectors is $\vec{r}(t)$, where $\vec{r}(t)$ might be defined as follows:

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$$

1.2.2 A Few Vector Operations

Definition 1.3. Inner Product: $\langle \mathbf{a}, \mathbf{b} \rangle$

The inner product (aka dot product or scalar product) of two n-dimensional vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, usually denoted by either $\langle \mathbf{a}, \mathbf{b} \rangle$ or $\mathbf{a} \cdot \mathbf{b}$, is defined to be the scalar value

$$\langle \mathbf{a}, \mathbf{b} \rangle := \sum_{j=1}^{n} a_j b_j \tag{2}$$

Since a_j and b_j are scalars (and scalar multiplication commutes) the inner product commutes:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} b_j a_j = \langle \mathbf{b}, \mathbf{a} \rangle \tag{3}$$

It is worth noting that there are some cases in which the dot notation is used but the operation does not commute. For example, the *divergence* of $\mathbf{A}$, is defined to be div $\mathbf{A} = \nabla \cdot \mathbf{A}$ [8]. Here the operation denoted by $\cdot$ is "reminiscent" of the inner product (as defined in Equation (2)) but does not commute. That is

[3] This is called a "one-hot" encoding in machine learning, where the i's might be the classes in a classification problem.
[4] A scalar function is a function f such that $f : \mathbb{R}^n \to \mathbb{R}$.

$$\nabla \cdot \mathbf{A} \neq \mathbf{A} \cdot \nabla \tag{4}$$

One way to see this is to notice that the LHS of Equation (4) is a scalar function while the RHS is a differential operator.

If $\mathbf{a}$ and $\mathbf{b}$ are column vectors then their inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ can also be written as the matrix product $\mathbf{a}^\intercal \mathbf{b}$. Similarly, if $\mathbf{a}$ and $\mathbf{b}$ are row vectors then their inner product can be written as the matrix product $\mathbf{a}\mathbf{b}^\intercal$. For example, the inner product of two $n \times 1$ column vectors $\mathbf{a}$ and $\mathbf{b}$ is equivalent to the following matrix multiplication:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\intercal \mathbf{b} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{j=1}^{n} a_j b_j$$

According to the Pythagorean theorem, the length of a vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, denoted $\|\mathbf{a}\|$, equals $\sqrt{a_1^2 + a_2^2 + a_3^2}$. The next definition is a generalization of the notion of length to vectors in $\mathbb{R}^n$.

Definition 1.4. Euclidean Norm: $\|\mathbf{x}\|$

The Euclidean norm of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is defined to be

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{j=1}^{n} x_j^2} \tag{5}$$

Definition 1.5. p-Norm: $\|\mathbf{x}\|_p$

The p-norm of a vector $\mathbf{x} = (x_1, x_2 \dots, x_n)$ is defined to be

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p \geq 1$ and $p \in \mathbb{R}$.

For $p = 1$, we get the taxicab norm, for $p = 2$ we get the Euclidean norm, and as p approaches ∞ the p-norm approaches the infinity norm or maximum norm: $\|\mathbf{x}\|_\infty := \max_i |x_i|$.

The inner product also has an interesting geometric interpretation:

Theorem 1.1. $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ are two linearly independent vectors in $\mathbb{R}^3$ and that M is the plane generated by them. Then $\mathbf{x}$ and $\mathbf{y}$ also generate a triangle in M with sides of length $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, and $\|\mathbf{x} - \mathbf{y}\|$. This setup is shown in Figure 1.

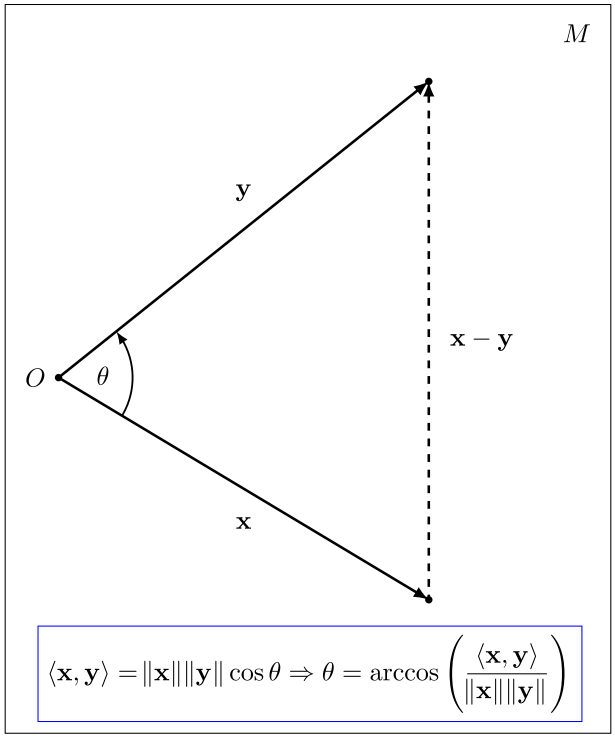

Figure 1: Setup for the geometric interpretation of $\langle \mathbf{x}, \mathbf{y} \rangle$

If $\theta \in \{0, \pi\}$ is the angle between $\mathbf{x}$ and $\mathbf{y}$ in M then by the Law of Cosines [10] we have

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta \tag{6}$$

We can also notice that

$$\begin{aligned}
\|\mathbf{x}-\mathbf{y}\|^2 &= \sum_{j=1}^{n} (x_j - y_j)^2 && \# \, \|\mathbf{x}\|^2 = \sum_{j=1}^{n} x_j^2 \text{ (Equation (5))} \\
&= \sum_{j=1}^{n} \left(x_i^2 - 2x_j y_j + y_j^2\right) && \# \, (x_j - y_j)^2 = x_i^2 - 2x_j y_j + y_j^2 \\
&= \sum_{j=1}^{n} x_j^2 - 2\sum_{j=1}^{n} x_j y_j + \sum_{j=1}^{n} y_j && \# \text{ sum is a linear operator [12]} \\
&= \|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 && \# \, \|\mathbf{x}\|^2 = \sum_{j=1}^{n} x_j^2 \text{ and } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j \text{ (Equations (5) and (2))} \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle && \# \text{ rearrange: } \|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

Setting the right-and side (RHS) of this last expression for $\|\mathbf{x}-\mathbf{y}\|^2$ equal to the RHS of Equation (6) yields

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$$

Subtracting $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ from both sides of this equation and then dividing by -2 gives us

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$$

Definition 1.6. Orthogonal Vectors: $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

We say that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, since

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta && \# \text{ Theorem 1.1} \\
&\Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos\tfrac{\pi}{2} && \# \ \mathbf{x} \perp \mathbf{y} \Rightarrow \theta = \tfrac{\pi}{2} \\
&\Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \, 0 && \# \cos\tfrac{\pi}{2} = 0 \\
&\Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle = 0 && \# \, \|\mathbf{x}\|\|\mathbf{y}\| \, 0 = 0
\end{aligned}$$

Definition 1.7. Orthogonal Projection

Given two linearly independent vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we want to find the orthogonal projection of $\mathbf{a}$ onto the line generated by $\mathbf{b}$. To analyze this, denote by M the plane generated by $\mathbf{a}$ and $\mathbf{b}$ and consider an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of M consisting of $\mathbf{v}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|}$ ($\mathbf{v}_1 = \hat{\mathbf{b}}$) and a unit vector $\mathbf{v}_2 \in M$ that is orthogonal to $\mathbf{v}_1$. This is depicted in Figure 2.

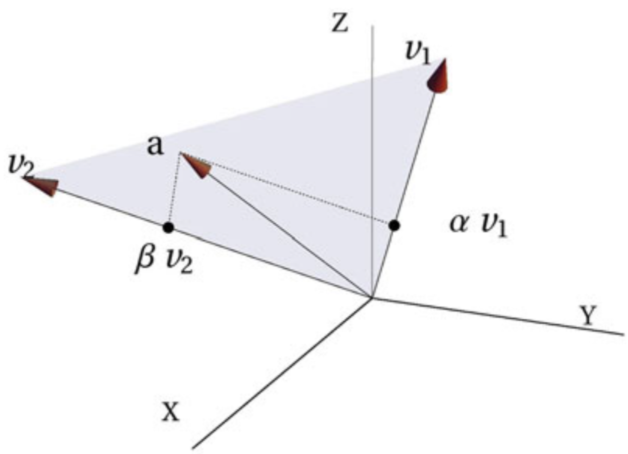

Figure 2: Orthogonal Projection

Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of M there exist scalars α and β such that

$$\mathbf{a} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \tag{7}$$

Here we want to solve for α. One way to do that is to take the inner product of $\mathbf{b}$ and $\mathbf{a}$ (Equation (7)):

$$
\begin{aligned}
\langle \mathbf{a}, \mathbf{b} \rangle &= \langle \mathbf{b}, \mathbf{a} \rangle && \text{\# since } \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} b_j a_j = \langle \mathbf{b}, \mathbf{a} \rangle \text{ (Equation (3))} \\
&= \langle \mathbf{b}, (\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) \rangle && \text{\# since } \mathbf{a} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \text{ (Equation (7))} \\
&= \langle \mathbf{b}, \alpha \mathbf{v}_1 \rangle + \langle \mathbf{b}, \beta \mathbf{v}_2 \rangle && \text{\# the inner product is distributive over vector addition [15]} \\
&= \alpha \langle \mathbf{b}, \mathbf{v}_1 \rangle + \beta \langle \mathbf{b}, \mathbf{v}_2 \rangle && \text{\# by the scalar product rule: } \langle c_1 \mathbf{v}_1, c_2 \mathbf{v}_2 \rangle = c_1 c_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \text{ [15]} \\
&= \alpha \langle \mathbf{v}_1, \mathbf{b} \rangle + \beta \langle \mathbf{b}, \mathbf{v}_2 \rangle && \text{\# since } \langle \mathbf{b}, \mathbf{v}_1 \rangle = \langle \mathbf{v}_1, \mathbf{b} \rangle \text{ (Equation (3))} \\
&= \alpha \langle \mathbf{v}_1, \mathbf{b} \rangle + \beta\, 0 && \text{\# since } \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0 \text{ and } \mathbf{v}_1 = \hat{\mathbf{b}} \Rightarrow \langle \hat{\mathbf{b}}, \mathbf{v}_2 \rangle = \tfrac{1}{\|\mathbf{b}\|} \langle \mathbf{b}, \mathbf{v}_2 \rangle = 0 \Rightarrow \langle \mathbf{b}, \mathbf{v}_2 \rangle = 0 \\
&= \alpha \langle \mathbf{v}_1, \mathbf{v_1} \|\mathbf{b}\| \rangle && \text{\# since } \beta\, 0 = 0 \text{ and } \mathbf{v}_1 = \tfrac{\mathbf{b}}{\|\mathbf{b}\|} \text{ so } \mathbf{b} = \mathbf{v_1} \|\mathbf{b}\| \\
&= \alpha \|\mathbf{b}\| \, \langle \mathbf{v}_1, \mathbf{v}_1 \rangle && \text{\# by the scalar product rule} \\
&= \alpha \|\mathbf{b}\| && \text{\# since } \mathbf{v}_1 = \tfrac{\mathbf{b}}{\|\mathbf{b}\|} \Rightarrow \|\mathbf{v}_1\| = 1 \text{ and so } \langle \mathbf{v}_1, \mathbf{v_1} \rangle = \|\mathbf{v}_1\|^2 = 1^2 = 1
\end{aligned}
$$

So we see that

$$\langle \mathbf{a}, \mathbf{b} \rangle = \alpha \|\mathbf{b}\|$$

and so

$$\alpha = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{b}\|} = \left\langle \mathbf{a}, \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\rangle = \langle \mathbf{a}, \hat{\mathbf{b}} \rangle$$

α represents the component of the vector $\mathbf{a}$ that is parallel to ("in the direction of") the vector $\hat{\mathbf{b}}$. More generally, since $\hat{\mathbf{b}}$ is the unit vector in the $\mathbf{b}$ direction, $\langle \mathbf{a}, \mathbf{b} \rangle$ is also the component of $\mathbf{a}$ in the $\mathbf{b}$ direction.

Definition 1.8. Cross Product: $\mathbf{a} \times \mathbf{b}$

The cross product is a binary operation on two vectors in a three-dimensional oriented Euclidean vector space [5]. Specifically, given two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \times \mathbf{b}$, is defined to be a vector with the following three properties:

- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (Figure 3).
- $\mathbf{a} \times \mathbf{b}$ has direction given by the right-hand rule (Figure 4).
- $\mathbf{a} \times \mathbf{b}$ has magnitude equal to the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ (Figure 3).

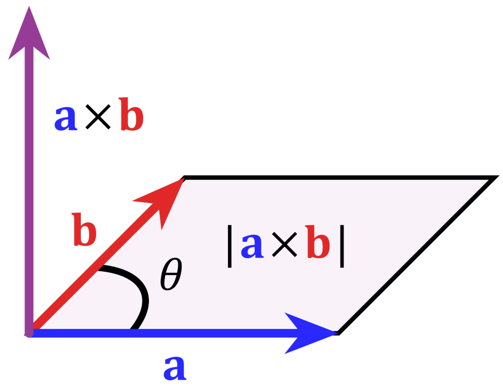

Figure 3: The parallelogram formed by $\mathbf{a} \times \mathbf{b}$ [14]

One definition of the cross product is

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta)\, \mathbf{n} \tag{8}$$

where

- θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ in the plane containing them, so $0 \leq \theta \leq \pi$.
- $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ are the magnitudes of the vectors $\mathbf{a}$ and $\mathbf{b}$ (aka their norms; see Definition 1.4)).
- $\mathbf{n}$ is a unit vector normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in the direction given by the right-hand rule.

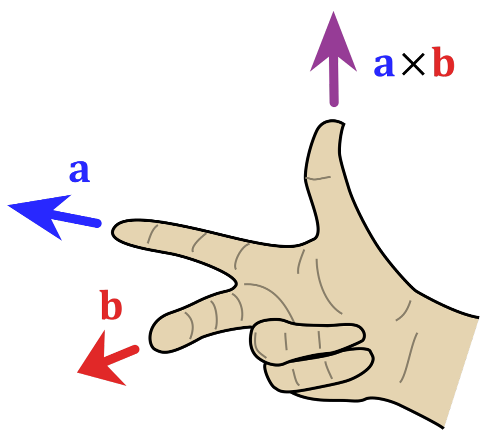

Figure 4: The right-hand rule for vectors $\mathbf{a}$ and $\mathbf{b}$ [17]

If the cross product of two vectors is the zero vector (that is, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$), then either one or both of the inputs is the zero vector ($\mathbf{a} = \mathbf{0}$ and/or $\mathbf{b} = \mathbf{0}$) or the vectors are parallel or anti-parallel ($\theta \in \{0, \pi\}$). In the second case ($\theta \in \{0, \pi\}$), since the angle between a vector and itself is zero, Equation (8) tells us that the cross product of a vector with itself is the zero vector: $\mathbf{a} \times \mathbf{a} = \|\mathbf{a}\| \|\mathbf{a}\| \sin(0)\, \mathbf{n} = \|\mathbf{a}\| \|\mathbf{a}\|\, 0\, \mathbf{n} = \mathbf{0}$.

Note that by the right-hand rule, if the thumb points at you then the fingers curl in the anti-clockwise (counter-clockwise) direction. That is, when viewed from the top or z axis the system rotates in an anti-clockwise direction. This is one of the reasons we take the direction of a curve to be in the anti-clockwise direction when we consider, for example, line integrals in Section 6.

The cross product can also be expressed as the following determinant

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

which can be computed using Sarrus's rule [13] or cofactor expansion [3]. Using Sarrus's rule the cross product expands to

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_2 b_3 \hat{\mathbf{i}} + a_3 b_1 \hat{\mathbf{j}} + a_1 b_2 \hat{\mathbf{k}}) - (a_3 b_2 \hat{\mathbf{i}} + a_1 b_3 \hat{\mathbf{j}} + a_2 b_1 \hat{\mathbf{k}}) \\ &= (a_2 b_3 - a_3 b_2)\hat{\mathbf{i}} + (a_3 b_1 - a_1 b_3)\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{k}} \end{aligned}$$

Using cofactor expansion the 3×3 determinant can be expressed in terms of 2×2 determinants:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\mathbf{k}} \\ &= (a_2 b_3 - a_3 b_2)\hat{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{k}} \end{aligned}$$

The cross product has the following additional properties [14]:

- It is anti-commutative: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- It is distributive over addition: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
- It is compatible with scalar multiplication so that: $(c\,\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (c\,\mathbf{b}) = c\,(\mathbf{a} \times \mathbf{b})$
- It is not associative but satisfies the Jacobi identity [16]: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

2 A Few Theorems

Theorem 2.1. Cauchy–Schwarz Inequality: $\left|\langle \mathbf{a}, \mathbf{b} \rangle\right| \leq \|\mathbf{a}\| \|\mathbf{b}\|$

Proof. The inequality is trivial if either $\mathbf{a}$ or $\mathbf{b}$ is zero, so assume that neither is. So without loss of generality (I think) let $\mathbf{x} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ and $\mathbf{y} = \dfrac{\mathbf{b}}{\|\mathbf{b}\|}$ and so $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ ($\mathbf{x} = \hat{\mathbf{a}}$ and $\mathbf{y} = \hat{\mathbf{b}}$). Then

$$
\begin{aligned}
0 &\leq \|\mathbf{x}-\mathbf{y}\| && \#\ \|\mathbf{x}-\mathbf{y}\| \geq 0 \text{ for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^n \text{ (Equation (5))} \\
&\leq \|\mathbf{x}-\mathbf{y}\|^2 && \#\ \|\mathbf{x}-\mathbf{y}\| \geq 0 \Rightarrow \|\mathbf{x}-\mathbf{y}\|^2 \geq 0 \\
&\leq \sum_{j=1}^{n}(x_j-y_j)^2 && \#\ \|\mathbf{x}-\mathbf{y}\|^2 = \sum_{j=1}^{n}(x_j-y_j)^2 \text{ (Equation (5))} \\
&\leq \sum_{j=1}^{n}\left(x_j^2-2x_jy_j+y_j^2\right) && \#\ \text{expand previous line} \\
&\leq \sum_{j=1}^{n}x_j^2-2\sum_{j=1}^{n}x_jy_j+\sum_{j=1}^{n}y_j^2 && \#\ \text{sum is a linear operator} \\
&\leq \|\mathbf{x}\|^2-2\langle\mathbf{x},\mathbf{y}\rangle+\|\mathbf{y}\|^2 && \#\ \|\mathbf{x}\|^2=\sum_{j=1}^{n}x_j^2 \text{ and } \langle\mathbf{x},\mathbf{y}\rangle=\sum_{j=1}^{n}x_jy_j \text{ (Equations (5) and (2) respectively)} \\
&\leq 1-2\langle\mathbf{x},\mathbf{y}\rangle+1 && \#\ \|\mathbf{x}\|=\|\mathbf{y}\|=1 \Rightarrow \|\mathbf{x}\|^2=\|\mathbf{y}\|^2=1 \\
&\leq 2-2\langle\mathbf{x},\mathbf{y}\rangle && \#\ 1+1=2 \\
&\Rightarrow -2 \leq -2\langle\mathbf{x},\mathbf{y}\rangle && \#\ \text{subtract 2 from both sides} \\
&\Rightarrow 1 \geq \langle\mathbf{x},\mathbf{y}\rangle && \#\ \text{divide by -2, reversing inequality} \\
&\Rightarrow \langle\mathbf{x},\mathbf{y}\rangle \leq 1 && \#\ \text{rearrange} \\
&\Rightarrow \left\langle\frac{\mathbf{a}}{\|\mathbf{a}\|},\frac{\mathbf{b}}{\|\mathbf{b}\|}\right\rangle \leq 1 && \#\ \mathbf{x}=\frac{\mathbf{a}}{\|\mathbf{a}\|},\ \mathbf{y}=\frac{\mathbf{b}}{\|\mathbf{b}\|} \text{ by the definition above} \\
&\Rightarrow \frac{\langle\mathbf{a},\mathbf{b}\rangle}{\|\mathbf{a}\|\|\mathbf{b}\|} \leq 1 && \#\ \text{scalar multiplication rule: } \langle c_1\mathbf{v}_1,c_2\mathbf{v}_2\rangle=c_1c_2\langle\mathbf{v}_1,\mathbf{v}_2\rangle \\
&\Rightarrow \langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\| && \#\ \text{multiply both sides by } \|\mathbf{a}\|\|\mathbf{b}\|
\end{aligned}
$$

If we replace $\mathbf{a}$ by $-\mathbf{a}$ we see by Equation (5) that $\|-\mathbf{a}\| = \|\mathbf{a}\|$. Further, by the scalar multiplication rule for inner products with $c_1 = -1$ and $c_2 = 1$ we have $\langle-\mathbf{a},\mathbf{b}\rangle = -\langle\mathbf{a},\mathbf{b}\rangle$. Taken together these results imply that $-\langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\|$.

Next note that the absolute value of x, $|x|$, is defined to be

$$
|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases} \tag{9}
$$

So one way to see the Cauchy–Schwarz inequality is by setting x equal to $\langle\mathbf{a},\mathbf{b}\rangle$ in Equation (9). Then when $\langle\mathbf{a},\mathbf{b}\rangle \geq 0$ we have $\big|\langle\mathbf{a},\mathbf{b}\rangle\big| = \langle\mathbf{a},\mathbf{b}\rangle$ and we showed above that $\langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\|$. Similarly, when $\langle\mathbf{a},\mathbf{b}\rangle < 0$ we have $\big|\langle\mathbf{a},\mathbf{b}\rangle\big| = -\langle\mathbf{a},\mathbf{b}\rangle$ and we showed that $-\langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\|$. Together these give $\big|\langle\mathbf{a},\mathbf{b}\rangle\big| \leq \|\mathbf{a}\|\|\mathbf{b}\|$. Said another way: $\Big[\big(-\langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\|\big) \wedge \big(\langle\mathbf{a},\mathbf{b}\rangle \leq \|\mathbf{a}\|\|\mathbf{b}\|\big)\Big] \Rightarrow \big|\langle\mathbf{a},\mathbf{b}\rangle\big| \leq \|\mathbf{a}\|\|\mathbf{b}\|$. ■

3 Derivatives

We will also denote the derivative of a vector $\mathbf{r}$ with respect to t by $\dot{\mathbf{r}}$, $\frac{d\mathbf{r}}{dt}$ or $\mathbf{r}'(t)$. Note also that by definition

$$
\frac{d\mathbf{r}}{dt} := \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}
$$

4 The Jacobian

It is common to change the variable(s) of integration with the main goal being to rewrite a complicated integrand into a some simpler but equivalent form. However, in doing so, the underlying geometry of the problem may be altered. This was seen often in single-variable integrals:

Example 4.1. Evaluate

$$\int_2^4 (3x+1)^3 dx \tag{10}$$

Solution: We can rewrite the integral in Equation (10) by letting $u = 3x+1$ so that $du = 3\,dx$. Notice that the expression $du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$ and so the bounds of integration in the xy-plane, namely $2 \leq x \leq 4$ are transformed using $u = 3x+1$. So if $x = 2$ then $u = 3(2)+1 = 7$ and when $x = 4$ we have $u = 3(4)+1 = 13$. So the bounds of integration with respect to u are $7 \leq u \leq 13$. Using this substitution we see that

$$\int_2^4 (3x+1)^3 dx = \int_7^{13} u^3 \left(\frac{1}{3}\,du\right) = \frac{1}{3}\int_7^{13} u^3\,du$$

Note that integrals $\displaystyle\int_2^4 (3x+1)^3 dx$ and $\displaystyle\frac{1}{3}\int_7^{13} u^3\,du$ represent the same problem. Specifically

$$\int_2^4 (3x+1)^3 dx = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left((3x+1)^4\right)\Big|_2^4 = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(13^4 - 7^4\right) = \left(\frac{1}{12}\right)26160 = 2180$$

and

$$\frac{1}{3}\int_7^{13} u^3\,du = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(u^4\right)\Big|_7^{13} = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(13^4 - 7^4\right) = \left(\frac{1}{12}\right)26160 = 2180$$

The $\frac{1}{3}$ that we see in the expression for dx, namely $dx = \frac{1}{3}\,du$ is called the (one dimensional) *Jacobian*.

Note also that the integral in variable x is over an interval of length 2 units, while the integral in u is over an interval of length 6 units. In a very rough sense then the variable u covers its interval of integration three times "as fast" as x. In particular, since u and x are linearly related the leading $\frac{1}{3}$ adjusts for the change in the underlying geometry of the intervals.

For double integrals in $\mathbb{R}^2$ we assume that a region of integration defined in terms of variables x and y and are substituted for new variables u and v through two functions:

$$\begin{aligned} u &= f_1(x,y) \\ v &= f_2(x,y) \end{aligned}$$

Note that the pair of equations are written so that u and v are written in terms of x and y. This is called a *transformation*. Such a "change of variables" should also be reversible. That is, we should be able to solve for x and y in terms of u and v:

$$\begin{aligned} x &= g_1(u,v) \\ y &= g_2(u,v) \end{aligned}$$

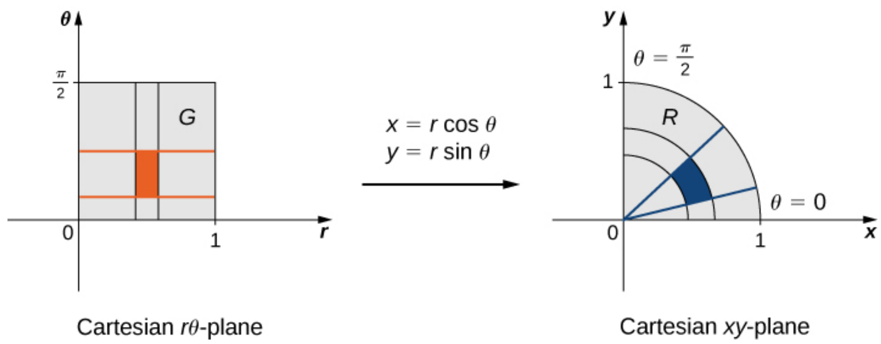

Figure 5: Transforming the $r\theta$-plane into the xy-plane

In this case the Jacobian, $J(u, v)$, is defined to be the determinant of a 2×2 matrix as follows:

$$J(u,v) = \det \begin{bmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\ \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$$

In three dimensions the Jacobian is

$$J(u,v,w) = \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

In general suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that each of its first-order partial derivatives exist on $\mathbb{R}^n$ This function takes a point $\mathbf{x} \in \mathbb{R}^n$ as input and produces the vector $f(x) \in \mathbb{R}^m$ as output. Then the Jacobian matrix of f is defined to be an $m \times n$ matrix J whose $(i, j)^{\text{th}}$ entry is

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j}$$

More explicitly

$$\mathbf{J} = \det \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \det \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $\nabla^{\mathrm{T}} f_i$ is the transpose (row vector) of the gradient of the i^{th} component.

I have also seen the Jacobian matrix, whose entries are functions of $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, denoted in various other ways including $\nabla \mathbf{f}$ and $\dfrac{\partial(f_1, .., f_m)}{\partial(x_1, .., x_n)}$.

Example 4.2. Find the Jacobian for the transformation shown in Figure 5.

Solution: Here we have

$$\begin{aligned}
J(r,\theta) &= \det\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} && \text{\# definition of the Jacobian} \\
&= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} && \text{\# } x = r\cos\theta, y = r\sin\theta \\
&= \cos(\theta) r\cos(\theta) - (-r\sin(\theta))\sin(\theta) && \text{\# } \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \\
&= r\cos^2\theta + r\sin^2\theta && \text{\# simplify} \\
&= r(\cos^2\theta + \sin^2\theta) && \text{\# factor out } r \\
&= r && \text{\# } \cos^2\theta + \sin^2\theta = 1
\end{aligned}$$

$J(r,\theta) = r$ is the common Jacobian when rectangular coordinates x and y are rewritten in polar coordinates r and θ.

For the next example we need the definition of Type I and Type II regions.

Definition 4.1. Type I and Type II Regions

A region D in the xy-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is: $D = \{(x,y)\,|\, a \le x \le b,\, g_1(x) \le y \le g_2(x)\}$.

Type I regions are shown in Figure 6.

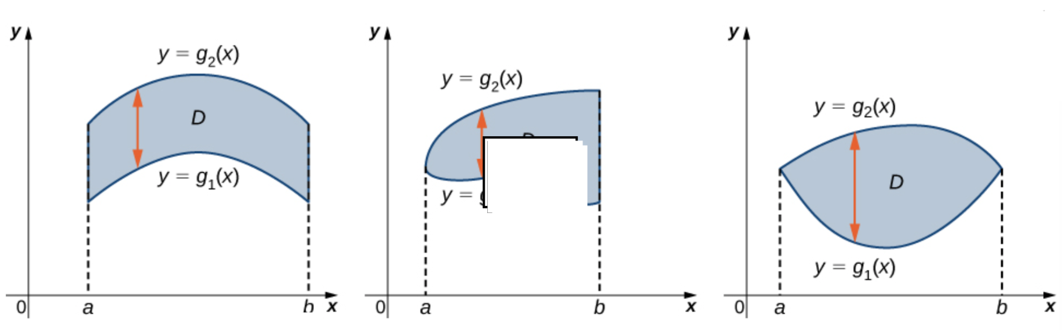

Figure 6: Type I Regions

Alternatively, a region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is

$$D = \{(x,y)\,|\, c \le y \le d,\, h_1(y) \le x \le h_2(y)\}$$

Type II regions are shown in Figure 7.

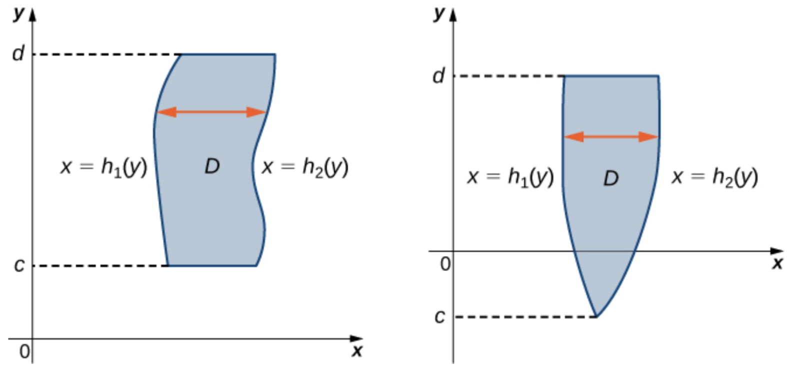

Figure 7: Type II Regions

Example 4.3. Evaluate $\iint_R (x - 2y)\, dA$, where R is a parallelogram in the xy-plane with vertices (0,0), (4,1), (6,4) and (2,3). This parallelogram is shown in Figure 8.

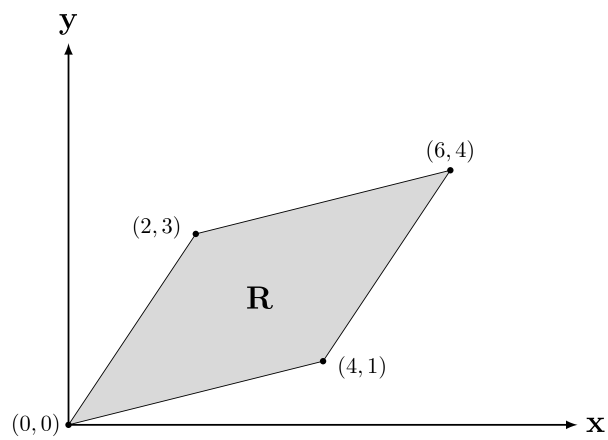

Figure 8: Parallelogram in the xy-plane

Solution: A sketch of this region (Figure 8) shows that it is a Type II region and would require three separate double integrals in either the dy, dx or $dxdy$ orderings of integration. Instead, note that the region consists of two pairs of parallel sides and so we can find the equation for each side:

- For the region from $(0, 0)$ to $(4, 1)$ we have $y = \frac{1}{4}x$ or $-x + 4y = 0$
- For the region from $(2, 3)$ to $(6, 4)$ we have $y = \frac{1}{4}x + \frac{5}{2}$ or $-x + 4y = 10$
- For the region from $(0, 0)$ to $(2, 3)$ we have $y = \frac{3}{2}x$ or $-\frac{3}{2}x$ or $-3x + 2y = 0$
- For the region from $(4, 1)$ to $(6, 4)$ we have $y = \frac{3}{2}x - 5$ or $-3x + 2y = -10$

This is shown in Figure 9.

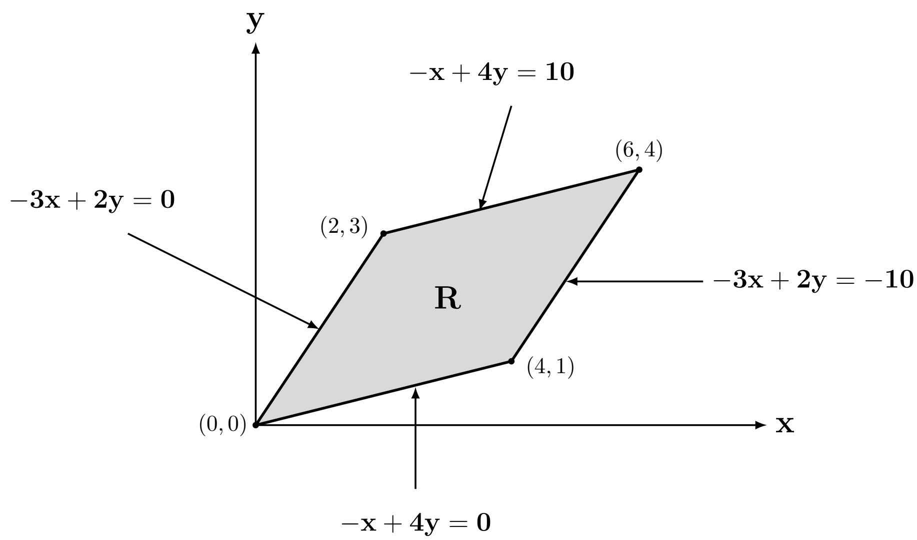

Figure 9: Equations of the sides of the parallelogram shown in Figure 8

Now we can define a transformation from x and y into the new variables u and v by the following equations:

$$u = -x + 4y \qquad \# \text{ so that } 0 \le u \le 10$$
$$v = -3x + 2y \qquad \# \text{ so that } -10 \le v \le 0$$

This transformation transforms the region of integration **R** in the xy-plane (a parallelogram, Figure 8) into a square in the uv-plane and so u and v have constant bounds. This square is shown Figure 10. The double-integral (whether written in xy-space or uv-space) should be negative because the integrand is "more often" negative over **R** and its transform into the uv-space.

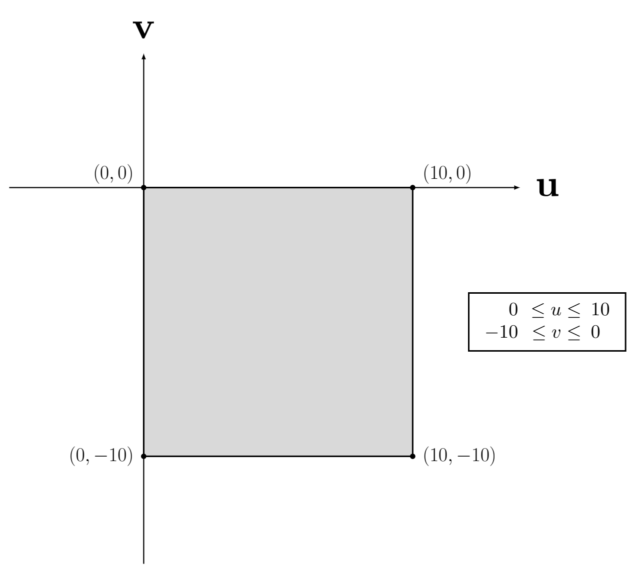

Figure 10: The parallelogram **R** (Figure 8) is a square in the uv-plane

Now we need to solve for x and y. We have expressions for u and v and so can solve for x and y:

$$u = -x + 4y$$
$$v = -3x + 2y$$

First multiply the bottom row by -2:

$$\begin{aligned} u &= -x + 4y \\ -2v &= -2(-3x + 2y) \end{aligned}$$

Simplifying we get

$$\begin{aligned} u &= -x + 4y \\ -2v &= 6x - 4y \end{aligned} \tag{11}$$

Adding these two equations we get $u - 2v = 5x$ and so solving for x we get

$$x = \frac{1}{5}u - \frac{2}{5}v \tag{12}$$

Substituting this back into the equation for u in Equations (11) and solving for y we get

$$y = \frac{3}{10}u - \frac{1}{10}v \tag{13}$$

We can now find the Jacobian:

$$\begin{aligned}
J(u,v) &= \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} && \text{\# definition of the Jacobian} \\
&= \det \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix} && \text{\# } x = \tfrac{1}{5}u - \tfrac{2}{5}v \text{ and } y = \tfrac{3}{10}u - \tfrac{1}{10}v \\
&= \left[\left(\frac{1}{5}\right)\left(-\frac{1}{10}\right)\right] - \left[\left(-\frac{2}{5}\right)\left(\frac{3}{10}\right)\right] && \text{\# } \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \\
&= -\frac{1}{50} + \frac{6}{50} && \text{\# simplify} \\
&= \frac{5}{50} && \text{\# simplify} \\
&= \frac{1}{10} && \text{\# } J(u,v) = \frac{1}{10}
\end{aligned}$$

So we see that the Jacobian in this case is

$$J(u,v) = \frac{1}{10} \tag{14}$$

Substituting Equations (12) and (13) into the original integral and noting that $dA = J(u,v)\,du\,dv$ we get

$$
\begin{aligned}
\iint\limits_R (x-2y)\,dA &= \int_{-10}^{0}\int_{0}^{10}\left(\left(\frac{1}{5}u-\frac{2}{5}v\right)-2\left(\frac{3}{10}u-\frac{1}{10}v\right)\right)\left(\frac{1}{10}\right)du\,dv && \text{\# Equations (12) \& (13) and } dA=\left(\tfrac{1}{10}\right)du\,dv \\
&= \frac{1}{10}\int_{-10}^{0}\int_{0}^{10}\left(-\frac{2}{5}u-\frac{1}{5}v\right)du\,dv && \text{\# simplify, moving the Jacobian to the front} \\
&= \frac{1}{10}\int_{-10}^{0}(-20-2v)\,dv && \text{\#} \int_0^{10}\left(-\frac{2}{5}u-\frac{1}{5}v\right)du=\left[-\frac{1}{5}u^2-\frac{1}{5}uv\right]\Bigg|_0^{10} \\
&= \frac{1}{10}\Big[-20v-v^2\Big]\Bigg|_{-10}^{0} && \text{\# evaluate integral + FToC} \\
&= 0-\frac{1}{10}\Big[-20(-10)-(-10)^2\Big] && \text{\# evaluate at limits} \\
&= \left(-\frac{1}{10}\right)(100) && \text{\#}\left(-\frac{1}{10}\right)(200-100)=-\frac{1}{10}(100) \\
&= -10 && \text{\# the region is completely below the } u \text{ axis} \\
\Rightarrow \left|\iint\limits_R (x-2y)\,dA\right| &= 10 && \text{\# the volume is 10 units}^3
\end{aligned}
$$

This was an example of a linear transformation, in which the equations transforming x and y into u and v were *linear* (so were the equations reversing the transformation). When this is the case the Jacobian will be a constant like we saw here.

We can also see how the geometry changed: The original region in the xy-plane has an area of 10 units^2 while the region in the uv-plane has an area of 100 units^2. That is, the region in the uv-plane is 10 times as large as the region in the xy-plane. The Jacobian $\left(\frac{1}{10}\right)$ scales this change in the underlying area.

Remark 4.1. Note that the Jacobian is usually taken to be a positive quantity. This is because the naming (and ordering) of the functions transforming x and y into u and v (and the reverse) is arbitrary. Since the Jacobian is a determinant, it is possible that two rows may be swapped depending on the original naming of the functions, which may introduce a factor of -1 into the result, which can be ignored.

5 Arc Length

Consider a segment of the parametric curve $\mathbf{r}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}} + k(t)\,\hat{\mathbf{k}}$ between two points $P = \mathbf{r}(t)$ and $Q = \mathbf{r}(t+\Delta t)$, as shown in Figure 11.

A question we might ask is: what is the length of the segment of the curve between P and Q? This length is called the arc length and is denoted by Δs and can be approximated by the chord length $\|\Delta\mathbf{r}\|$.

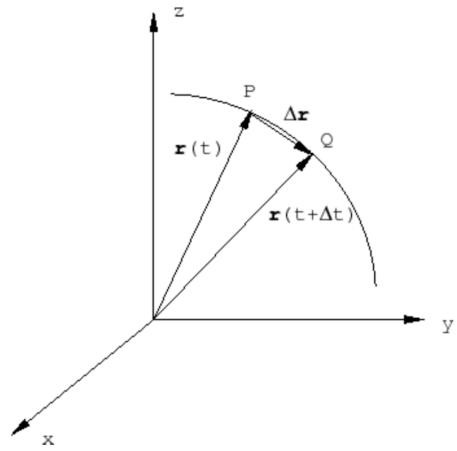

Figure 11: $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$

Specifically we can see from Figure 11 that

$$\|\Delta r\| = \|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|$$

and so

$$\Delta s \approx \|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|$$

Now we can compute the infinitesimal arc length ds, as shown in Figure 12:

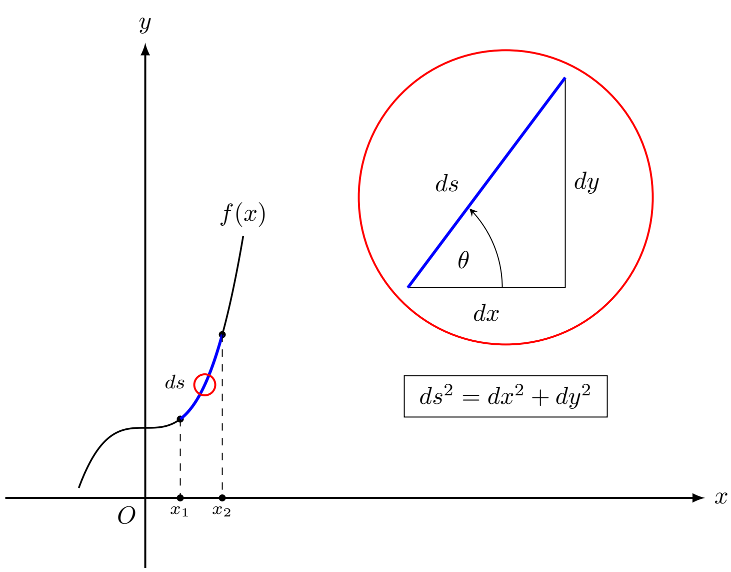

Figure 12: $f(x)$, ds and the Pythagorean Theorem

$$
\begin{aligned}
\Delta s &\approx \|\Delta \mathbf{r}\| && \#\text{ approximate } \Delta s \text{ by } \|\Delta \mathbf{r}\| \\
&= \|\mathbf{r}(t+\Delta t) - \mathbf{r}(t)\| && \#\ \Delta \mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t) \\
&= \|\mathbf{r}(t+\Delta t) - \mathbf{r}(t)\| \cdot \frac{\Delta t}{\Delta t} && \#\text{ multiply by } 1 = \frac{\Delta t}{\Delta t} \\
&= \left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \Delta t && \#\text{ assume } \Delta t > 0 \\
&\Rightarrow \ \Delta s \approx \left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \Delta t && \#\text{ combine expressions for } \Delta s \\
&\Rightarrow \ \lim_{\Delta t \to 0} \Delta s = \lim_{\Delta t \to 0} \left[\left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \Delta t \right] && \#\text{ take the limit of both sides} \\
&\Rightarrow \ ds = \lim_{\Delta t \to 0} \left[\left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \Delta t \right] && \#\ \Delta t \to 0 \Rightarrow \Delta s \to 0 \Rightarrow \lim_{\Delta t \to 0} \Delta s = ds \\
&\Rightarrow \ ds = \lim_{\Delta t \to 0} \left\| \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \lim_{\Delta t \to 0} \Delta t && \#\text{ product rule for limits [1]} \\
&\Rightarrow \ ds = \left\| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right\| \cdot \lim_{\Delta t \to 0} \Delta t && \#\text{ for a normed vector space } X \ \lim \|x_n\| = \|\lim x_n\| \\
&\Rightarrow \ ds = \left\| \frac{d\mathbf{r}}{dt} \right\| \cdot dt && \#\ \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt},\ \lim_{\Delta t \to 0} \Delta t = dt
\end{aligned}
$$

Interestingly we can also see this using the Pythagorean theorem. Specifically, if we move an infinitesimal distance in the x and y directions we get a triangle in the xy-plane who's sides have length dx and dy and who's hypotenuse is ds. This scenario is shown for some curve $f(x)$ in Figure 12. In the two dimensional case where the parameterization is $x = g(t)$ and $y = h(t)$ we define $\mathbf{r}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}}$ so that

$$
\frac{d\mathbf{r}}{dt} = \frac{dg}{dt}\,\hat{\mathbf{i}} + \frac{dh}{dt}\,\hat{\mathbf{j}} \tag{15}
$$

The Pythagorean theorem tells us that $ds = \sqrt{dx^2 + dy^2}$. Using this expression for ds we can see that

$$
\begin{aligned}
ds &= \sqrt{dx^2 + dy^2} && \text{\# Figure 12} \\
&= \sqrt{dg^2 + dh^2} && \text{\# switch to parametric form: } x = g(t) \text{ and } y = h(t) \\
&= \sqrt{dg^2 + dh^2} \cdot \frac{dt}{dt} && \text{\# multiply by } 1 = \frac{dt}{dt},\ dt > 0 \\
&= \sqrt{dg^2 + dh^2} \cdot \left[\sqrt{\frac{1}{dt^2}} \cdot dt\right] && \text{\# } \frac{dt}{dt} = \sqrt{\left(\frac{dt}{dt}\right)^2} = \sqrt{\frac{dt^2}{dt^2}} = \sqrt{\frac{1}{dt^2} \cdot dt^2} = \sqrt{\frac{1}{dt^2}} \cdot dt = \left[\sqrt{\frac{1}{dt^2}} \cdot dt\right] \\
&= \sqrt{\left(dg^2 + dh^2\right) \cdot \left(\frac{1}{dt^2}\right)} \cdot dt && \text{\# simplify} \\
&= \sqrt{\left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} \cdot dt && \text{\# simplify} \\
&= \left\|\frac{d\mathbf{r}}{dt}\right\| \cdot dt && \text{\# Equation (15) and the definition of } \|\cdot\|
\end{aligned}
$$

Now we can see that

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\|$$

and so

$$\frac{ds}{dt} = \|\dot{\mathbf{r}}\| \tag{16}$$

5.1 The Unit Tangent Vector T

Since $\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \|\dot{\mathbf{r}}\|$ we also know that the unit tangent vector $\mathbf{T} = \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} = \frac{d\mathbf{r}}{ds}$ [7]. Why? Consider

$$
\begin{aligned}
\mathbf{T} &= \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} && \text{\# } \mathbf{T} \text{ is the unit vector in the } \dot{\mathbf{r}} \text{ direction (Equation (1))} \\
&= \frac{\left(\frac{d\mathbf{r}}{dt}\right)}{\|\dot{\mathbf{r}}\|} && \text{\# } \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} \text{ (definition of } \dot{\mathbf{r}}) \\
&= \frac{\left(\frac{d\mathbf{r}}{dt}\right)}{\left(\frac{ds}{dt}\right)} && \text{\# } \|\dot{\mathbf{r}}\| = \frac{ds}{dt} \text{ (Equation (16))} \\
&= \frac{d\mathbf{r}}{ds} && \text{\# simplify}
\end{aligned}
$$

6 The Line Integral

Consider the definite integral shown in Figure 13. Here we want to find the area **S** that is above the line segment $[a, b]$ and below the curve $f(x)$.

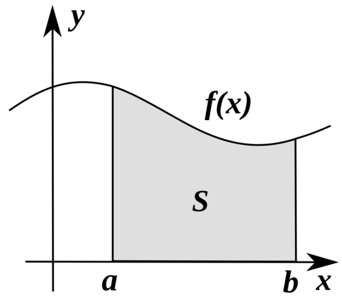

Figure 13: The Definite Integral: The area $\mathbf{S} = \int_a^b f(x)\,dx$

The line integral, denoted $\int_C f(x,y)\,ds$, is similar except that here we want to find the area above the curve C and below the function $f(x,y)$ (so the line integral is by definition in three dimensions). The comparison between the definite integral $\int_a^b f(x)\,dx$ and the line integral $\int_C f(x,y)\,ds$ is shown in Figure 14.

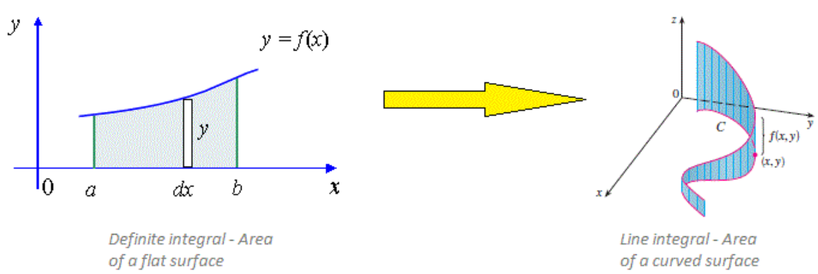

Figure 14: Definite vs. Line Integrals

6.0.1 Parameterizing the Curve C

One of the first steps in solving a line integral is to find a parameterization for the curve C. We would like to find a parameterization such that the integral reduces to an integral over a single variable, call it t. Then the parametric form of x is called $g(t)$ (or sometimes $x(t)$), the parametric form of y is called $h(t)$ (or $y(t)$), and the parameter $t \in [a, b]$. That is

$$\int_C f(x,y)\,ds = \int_a^b f(g(t), h(t))\,ds$$

All good, but what is ds? We saw in Section 5 that ds is the infinitesimal arc length and that by the Pythagorean theorem $ds = \sqrt{dx^2 + dy^2}$ (Figure 12). Using this expression for ds we can see that

$$
\begin{aligned}
ds &= \sqrt{dx^2 + dy^2} && \text{\# by the Pythagorean theorem} \\
&= \sqrt{dg^2 + dh^2} && \text{\# switch to parametric form: } x = g(t) \text{ and } y = h(t) \\
&= \sqrt{dg^2 + dh^2}\,\frac{dt}{dt} && \text{\# multiply by } 1 = \frac{dt}{dt},\ dt > 0 \\
&= \sqrt{\left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\,dt && \text{\# simplify} \\
&= \sqrt{g'(t)^2 + h'(t)^2}\,dt && \text{\# switch notation from Leibniz} \to \text{Lagrange: } \frac{df}{dt} = f'(t)
\end{aligned}
$$

Now we can write the line integral in terms of a single variable $t \in [a, b]$ as follows:

$$
\int_C f(x, y)\,ds = \int_a^b f(g(t), h(t))\sqrt{g'(t)^2 + h'(t)^2}\,dt
$$

For example, suppose C is a circle of radius r where we want to integrate over the part of the circle in the first quadrant. That is, C is $x^2 + y^2 = r^2$. In this case the parameterization is $x(t) = g(t) = r\cos(t)$ and $y(t) = h(t) = r\sin(t)$. Solving for $\sqrt{g'(t)^2 + h'(t)^2}$ we get

$$
\begin{aligned}
\sqrt{g'(t)^2 + h'(t)^2} &= \sqrt{(-r\sin(t))^2 + (r\cos(t))^2} && \text{\# } g'(t) = -r\sin(t) \text{ and } h'(t) = r\cos(t) \\
&= \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} && \text{\# squares} \\
&= \sqrt{r^2(\sin^2(t) + \cos^2(t))} && \text{\# factor out } r^2 \\
&= r\sqrt{\sin^2(t) + \cos^2(t)} && \text{\# } \sqrt{r^2} = r \\
&= r\sqrt{1} && \text{\# } \sin^2(t) + \cos^2(t) = 1 \\
&= r && \text{\# } \sqrt{1} = 1
\end{aligned}
$$

So in this example $ds = r\,dt$.

Next we need to specify the limits of integration for the parameterization. Since the curve C is a circle in the first quadrant the parameter $t \in [0, \frac{\pi}{2}]$ and so the limits of integration are $a = 0$ and $b = \frac{\pi}{2}$. Putting this all together we get

$$
\int_0^{\frac{\pi}{2}} f(r\cos(t), r\sin(t))\,r\,dt = r\int_0^{\frac{\pi}{2}} f(r\cos(t), r\sin(t))\,dt
$$

for some function f.

6.1 Vector Fields

In vector calculus and physics, a vector field is an assignment of a vector to each point in a subset of some space [6]. For example, a vector field in the plane can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane. Vector fields are often used to model the speed anddirection of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from one point to another point.

For example, the wind velocity vector field for the 2011 Joplin, MO tornado [11] is shown in Figure 15. Here the color represents the wind speed $\|\mathbf{v}\|$, where $\mathbf{v}$ is the wind velocity vector.

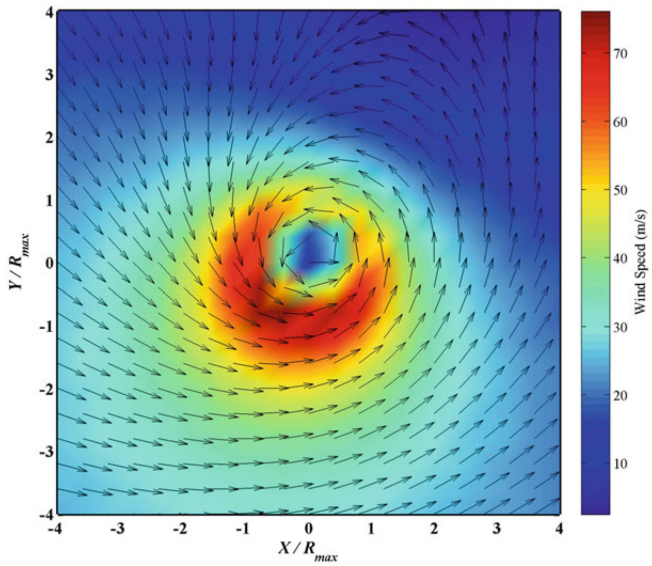

Figure 15: 2011 Joplin, MO Tornado Wind Velocity Vector Field

The general form of a vector field (here in three dimensions) is

$$\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$$

where P, Q, and R are scalar functions.

6.2 Line Integrals of Vector Fields

6.3 Fundamental Theorem for Line Integrals

6.4 Conservative Vector Fields

6.5 Green's Theorem

6.6 Stoke's Theorem

7 Surface Integrals

8 Conclusions

Acknowledgements

Thanks to Dave Neary who pointed out that using the reverse triangle inequality in my proof of the Cauchy–Schwarz inequality was overkill. In particular, $\|\mathbf{x} - \mathbf{y}\| \geq 0$ for all $\mathbf{x}$ and $\mathbf{y}$ by Equation (5), so the triangle inequality (or reverse triangle inequality) was not needed.

Thanks also to Ben Reiniger (@bmreiniger@mathstodon.xyz) for pointing out that in Figure 10 we care about whether the integrand is positive/negative over the region, but the region itself is just an area in the plane.

Appendix A

A Brief Review of Algebraic Structures

Structure	**ABO**[1]	**Identity**	**Inverse**	**Distributive**[2]	**Commutative**[3]	**Comments**
Semigroup	✓	no	no	N/A	no	$(S, \circ)$
Monoid	✓	✓	no	N/A	no	Semigroup plus identity $\in S$
Group	✓	✓	✓	N/A	no	Monoid plus inverse $\in S$
Abelian Group	✓	✓	✓	N/A	✓$(\circ)$	Commutative group
Ring$_+$	✓	✓	✓	✓	✓$(+)$	Abelian group under $+$
Ring$_*$	✓	yes/no	no	✓	no	Monoid under $*$
Field$_{(+,*)}$	✓	✓$(+,*)$	✓$(+,*)$	✓	✓$(+,*)$	Abelian group under $+$ and $*$
Vector Space	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Field
Module	✓	✓$(+,*)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Ring

Table 1: A Few Algebraic Structures and Their Features

Abbreviations

1. **ABO:** Associative Binary Operation
   - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$
   - $x \circ y \in S$ for all $x, y \in S$ (S is closed under $\circ$)
2. **Distributive:** Distributive Property
   - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
   - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$

- $*$ is *distributive* over $+$ if $*$ is left and right distributive

3. **Commutative:** Commutative Property

   - $x \circ y = y \circ x$ for all $x, y \in S$

Notes

- Table 1 implies that F $\subset$ R $\subset$ G $\subset$ M $\subset$ SG.
- Whether or not a ring has a multiplicative identity seems to depend on the field of study.

  In general the definition of a ring R doesn't require a multiplicative inverse in R ($a^{-1} \notin R$ for all $a \in R$) or that multiplication be commutative in R. Specifically: R is an Abelian group under $+$ but we don't require that multiplication be commutative (while $a + b = b + a$ for all $a, b \in R$, we don't require that $ab = ba$ for all $a, b \in R$). These are perhaps the main ways in which a ring differs from a field. In addition, as mentioned above in some cases R need not include a multiplicative identity ($1 \notin R$).
- F $\subset$ VS since the field axioms require a multiplicative inverse (a^{-1}) while vector spaces do not. Fields are also commutative under $*$ and vector spaces are not.
- VS $\subset$ Module since the scalars in a module come from a ring as opposed to a field like we find in vector spaces and F $\subset$ R [2].

Appendix B

Fields and Vector Spaces

Fields

A *field* is an algebraic structure $\mathbb{K}$ in which we can add and multiply elements such that the following laws hold:

Addition Laws

(FA0) Closure: For any $a, b \in \mathbb{K}$ there is a unique element $a + b \in \mathbb{K}$.

(FA1) Associativity: For all $a, b, c \in \mathbb{K}$ we have $a + (b + c) = (a + b) + c$.

(FA2) Identity: There is an element $0 \in \mathbb{K}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{K}$.

(FA3) Inverse: For any $a \in \mathbb{K}$ there exists $-a \in \mathbb{K}$ such that $a + (-a) = (-a) + a = 0$.

(FA4) Commutativity: For any $a, b \in \mathbb{K}$ we have $a + b = b + a$.

Multiplication laws

(FM0) Closure: For any $a, b \in \mathbb{K}$, there is a unique element $ab \in \mathbb{K}$.

(FM1) Associativity: For all $a, b, c \in \mathbb{K}$ we have $a(bc) = (ab)c$.

(FM2) Identity: There is an element $1 \in \mathbb{K}$, $1 \neq 0$, such that $a1 = 1a = a$ for all $a \in \mathbb{K}$.

(FM3) Inverse: For any $a \in \mathbb{K}$ with $a \neq 0$, there exists $a^{-1} \in \mathbb{K}$ such that $aa^{-1} = a^{-1}a = 1$.

(FM4) Commutativity: For any $a, b \in \mathbb{K}$ we have $ab = ba$.

Distributive law

(D) Distributivity: For all $a, b, c \in \mathbb{K}$, we have $a(b + c) = ab + ac$.

Note the similarity of the addition and multiplication laws. We say that $(\mathbb{K}, +)$ is an *Abelian* group if (FA0)-(FA4) hold. (FM0)-(FM4) say that $(\mathbb{K}\backslash\{0\}, \cdot)$ is also an Abelian group (we have to leave out 0 because as (FM3) says, 0 does not have a multiplicative inverse).

Examples of fields include $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), and $\mathbb{Z}_p$ (the integers mod p, for p a prime number).

Associated with any field $\mathbb{K}$ is a non-negative integer called its characteristic, which is defined as follows: the characteristic of a field $\mathbb{K}$, often denoted char($\mathbb{K}$), is the smallest number of times one must use the field's (or ring's) multiplicative identity (1) in a sum to get the additive identity (0). If this sum never reaches the additive identity the field is said to have characteristic zero. That is,

$$\mathrm{char}(\mathbb{K}) = \begin{cases} n & n \text{ is the smallest positive number such that } \underbrace{1 + 1 + \cdots + 1}_{n} = 0 \\ 0 & \text{if the sum of ones never reaches } 0 \end{cases}$$

Important examples such as $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have characteristic zero, while $\mathbb{Z}_p$ has characteristic p (for prime p).

Vector Spaces

Let $\mathbb{K}$ be a field. A vector space V over $\mathbb{K}$ is an algebraic structure in which we can add two elements of V and multiply an element of V by an element of $\mathbb{K}$ (this is called *scalar multiplication*) such that the following rules hold:

Addition Laws

(VA0) Closure: For any $u, v \in V$ there is a unique element $u + v \in V$.

(VA1) Associativity: For all $u, v \in V$ we have $u + (v + w) = (u + v) + w$.

(VA2) Identity: There is an element $0 \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.

(VA3) Inverse: For any $v \in V$, there exists $-v \in V$ such that $v + (-v) = (-v) + v = 0$.

(VA4) Commutativity: For any $u, v \in V$ we have $u + v = v + u$.

Scalar multiplication laws

(VM0) Closure: For any $a \in \mathbb{K}$, $v \in V$ there is a unique element $av \in V$.

(VM1) Distributivity$_1$: For any $a \in \mathbb{K}$, $u, v \in V$ we have $a(u + v) = au + av$.

(VM2) Distributivity$_2$: For any $a, b \in \mathbb{K}$, $v \in V$ we have $(a + b)v = av + bv$.

(VM3) Associativity: For any $a, b \in \mathbb{K}$, $v \in V$ we have $(ab)v = a(bv)$.

(VM4) Identity: For any $v \in V$ we have $1v = v$ (where 1 is the element given by (FM2)).

Again, we can summarize (VA0)–(VA4) by saying that $(V, +)$ is an Abelian group.

One of the most important examples of a vector space over a field $\mathbb{K}$ is the set $\mathbb{K}^n$ of all n-tuples with elements from $\mathbb{K}$ (should prove that $\mathbb{K}^n$ is a vector space). Addition and scalar multiplication in $\mathbb{K}^n$ are defined by the following rules:

$$\begin{aligned}(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\ a(v_1, v_2, \ldots, v_n) &= (av_1, av_2, \ldots, av_n)\end{aligned}$$

Note that one of the key features of a vector space is closure under *componentwise* addition, as shown above.

Appendix C

I had thought that one way to prove the Cauchy–Schwarz inequality is to use the reverse triangle inequality, but Dave Neary pointed out that I didn't need it. So I'm leaving the derivation of the reverse triangle inequality in this appendix.

Theorem: Reverse Triangle Inequality: $\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\|$

We can derive the reverse triangle inequality from the triangle inequality [9, 18] by observing that

$$\begin{aligned}
\|\mathbf{x}\| &= \|\mathbf{x} + (-\mathbf{y} + \mathbf{y})\| && \text{\# add 0 to } \mathbf{x}\text{: } \mathbf{x} = \mathbf{x} + 0 = \mathbf{x} + (-\mathbf{y} + \mathbf{y}) \\
&= \|(\mathbf{x} - \mathbf{y}) + \mathbf{y}\| && \text{\# addition is associative: } \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 \\
&\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| && \text{\# by the triangle inequality: } \|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| \\
&\Rightarrow \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| && \text{\# combine expressions for } \|\mathbf{x}\| \\
&\Rightarrow \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| && \text{\# subtract } \|\mathbf{y}\| \text{ from both sides}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbf{y}\| &= \|\mathbf{y} + (-\mathbf{x} + \mathbf{x})\| && \text{\# add 0 to } \mathbf{y}\text{: } \mathbf{y} = \mathbf{y} + 0 = \mathbf{y} + (-\mathbf{x} + \mathbf{x}) \\
&= \|(\mathbf{y} - \mathbf{x}) + \mathbf{x}\| && \text{\# addition is associative} \\
&\leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\| && \text{\# by the triangle inequality} \\
&\Rightarrow \|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\| && \text{\# combine expressions for } \|\mathbf{y}\| \\
&\Rightarrow \|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| && \text{\# subtract } \|\mathbf{x}\| \text{ from both sides} \\
&\Rightarrow -\Big[\|\mathbf{y}\| - \|\mathbf{x}\|\Big] \geq -\|\mathbf{y} - \mathbf{x}\| && \text{\# multiply both sides by -1: } -1 * (\mathbf{v}_1 \leq \mathbf{v}_2) \Rightarrow -\mathbf{v}_1 \geq -\mathbf{v}_2 \\
&\Rightarrow \|\mathbf{x}\| - \|\mathbf{y}\| \geq -\|\mathbf{x} - \mathbf{y}\| && \text{\# } \|\mathbf{y} - \mathbf{x}\| = \|-(\mathbf{y} - \mathbf{x})\| = \|\mathbf{x} - \mathbf{y}\| \text{Equation (5)} \\
&\Rightarrow -\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\| && \text{\# rearrange}
\end{aligned}$$

Combining the expressions for $\|\mathbf{x}\| - \|\mathbf{y}\|$ we get $-\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$, that is, the reverse triangle inequality $\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\|$. ■

LaTeX Source

`https://www.overleaf.com/read/fgtfvmgdkbhh`

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