

# A Few Notes On The Riemann Zeta Function

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## 1 Introduction

TBD

### 1.1 Dirichlet Series

A Dirichlet Series [1] is an infinite series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1}$$

for  $s \in \mathbb{C}$ . Convention seems to be that it is unnecessary to note that  $n \in \mathbb{N}^+$ .

The complex variable  $s$  in Equation 1 is represented as  $s = \alpha + it$ . We use the notation  $\Re(s)$  and  $\Im(s)$  to indicate the real and imaginary parts of  $s$  respectively. That is,  $\alpha = \Re(s)$  and  $t = \Im(s)$ .

Perhaps the most famous example of a Dirichlet series is the Riemann zeta function  $\zeta(s)$ , where we take  $a_n = 1$  for all  $n \in \mathbb{N}^+$ :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

## 2 Euler's Product Formula

Recall that the Riemann zeta function (Equation 2) is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for  $s \in \mathbb{C}$ .

In 1737 Leonhard Euler [3] discovered the beautiful connection between the zeta function and the prime numbers by proving this identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \quad (3)$$

The left side of Equation 3 is by definition  $\zeta(s)$  (Equation 2). The infinite product on the right side of Equation 3 extends over all prime numbers  $p$  and is called a Euler Product [2]. A Euler Product is the expansion of a Dirichlet series (Equation 1) into an infinite product indexed by prime numbers:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdot \frac{1}{1 - 11^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots$$

The Euler Product Formula (and therefore the zeta function) converges for  $\Re(s) > 1$ .

## References

- [1] John E. McCarthy. Dirichlet Series. <https://www.math.wustl.edu/~mccarthy/amaster-ds.pdf>, 2018. [Online; accessed 25-June-2021].
- [2] Noam D. Elkies. Math 259: Introduction to Analytic Number Theory. <http://abel.math.harvard.edu/~elkies/M259.06/euler.pdf>, 1998. [Online; accessed 25-June-2021].
- [3] Timothy Murphy. Euler's Product Formula. <https://www.maths.tcd.ie/pub/Maths/Courseware/428/Primes-II.pdf>, 2006. [Online; accessed 25-June-2021].